

Verifiable Weighted Secret Sharing

Kareem Shehata 

School of Computing
National University of Singapore
kareem@comp.nus.edu.sg

Han Fangqi 

School of Computing
National University of Singapore
fhan4@comp.nus.edu.sg

Sri AravindaKrishnan Thyagarajan 

School of Computer Science
University of Sydney
aravind.thyagarajan@sydney.edu.au

Abstract—Traditionally, threshold secret sharing (TSS) schemes assume all parties have equal weight, yet emerging systems like blockchains reveal disparities in party trustworthiness, such as stake or reputation. Weighted Secret Sharing (WSS) addresses this by assigning varying weights to parties, ensuring security even if adversaries control parties with total weight at most a threshold t . Current WSS schemes assume honest dealers, resulting in security from only honest-but-curious behaviour but not protection from malicious adversaries for downstream applications. *Verifiable* secret sharing (VSS) is a well-known technique to address this, but existing VSS schemes are either tailored to TSS, or require additional trust assumptions. We propose the first efficient verifiable WSS scheme that tolerates malicious dealers and is compatible with the latest CRT-based WSS [9]. Our solution uses Bulletproofs for efficient verification and introduces new privacy-preserving techniques for proving relations between committed values, which may be of independent interest. Evaluation on Ethereum show up to a $100\times$ improvement in communication complexity compared to the current design and $20\times$ improvement compared to unweighted VSS schemes.

I. INTRODUCTION

Secret sharing is a fundamental cryptographic building block. In a secret-sharing scheme, a secret is distributed among a set of nodes such that only authorized subsets can cooperate to recover it. Many popular distributed protocols are built on secret sharing, including multi-party computation, threshold encryption, threshold signatures, and distributed randomness generation. The most well-known secret-sharing scheme is Shamir’s Secret Sharing (SSS) [14], which ensures that any set of corrupted parties up to a threshold t cannot learn the secret. A key assumption implicit in SSS and other threshold schemes is that all participating parties are of equal importance, trust, or weight.

In many recent applications, the notion of equal weight fails to capture real-world characteristics. For instance, in Proof-of-Stake (PoS) blockchains [10], each party is associated with a stake that can vary significantly. Similarly, in Oracle networks [3], servers are treated according to their respective reputation scores and an event is considered certified only if it is endorsed by a set of servers

This research is supported by Crystal Centre, National University of Singapore.

with enough total reputation score. Side-chains are another example where each party is associated with a “deposit” of the main chain’s currency, and the amount of the deposit determines the party’s voting power. These scenarios, among many others, highlight the necessity for schemes in which the adversary is modeled not by the number of individual nodes it corrupts, but by the total weight of the compromised nodes.

The traditional solution is to “virtualize” parties, replacing a party of weight w with w virtual parties of weight 1, and using an unweighted scheme such as SSS. This leads to communication and computation costs of $O(w \cdot \lambda)$ per party, which grows quickly in large systems with significant inequalities.

In [9], Garg et al. propose an efficient weighted secret sharing scheme based on the Chinese Remainder Theorem (CRT) with costs growing linearly with a party’s weight w , eliminating the λ factor. They describe a protocol with a trusted dealer and explore applications such as multi-party computation (MPC), threshold decryption, and threshold signatures. However, they only consider an honest dealer, consequently their applications can only tolerate semi-honest adversaries, i.e. parties are assumed to follow the protocol specification, but may try to learn additional information.

Verifiable secret sharing (VSS) [13] allows shareholders to verify the integrity of the deal, which is crucial when the dealer cannot be fully trusted, or when using VSS as a subprotocol. As stated in [9], upgrading their CRT-based scheme to VSS would enhance security to withstand malicious adversaries. While VSS is well-studied for linear schemes such as SSS, the only known CRT-based VSS [11] requires large unknown-order groups, relies on the strong RSA assumption, and needs a trusted setup. Since CRT-based secret sharing is non-linear, the techniques used by SSS-based VSS schemes cannot be applied and achieving an efficient CRT-based VSS without unknown-order groups has been unclear.

Therefore, we ask the question,

Can we construct an efficient CRT-based verifiable weighted secret sharing scheme without trusted setup, and assuming only a single group in which the discrete logarithm problem is hard?

In this paper, we answer the above question in the affirmative by enhancing the CRT-based weighted secret sharing scheme of [9] with verifiability of the shares in an efficient manner. We work with standard prime order elliptic curve groups (assuming the hardness of the discrete logarithm problem) which are widely used in real-world

arXiv:2505.24289v1 [cs.CR] 30 May 2025

applications, particularly in the case of blockchain systems where weighted secret sharing has become prominent.

A. Our Contributions

Our results in this work can be summarized as follows.

Verifiable Weighted Ramp Secret Sharing. We construct the first efficient non-interactive CRT-based verifiable weighted secret sharing scheme. Our scheme relies on prime order groups in which the discrete logarithm problem is assumed to be hard, such as an elliptic curve group, and does not require a trusted setup. Our verifiability technique applies to any CRT-based secret sharing scheme, including the threshold construction of [1] and the weighted-ramp¹ construction of [9]. The scheme is efficient, with communication cost scaling logarithmically in all parameters.

Proof-of-Mod. The technical core of our construction is a novel non-interactive zero-knowledge (NIZK) argument to prove that two committed values v, s satisfy $v = s \pmod p$, where p is a known prime. While this is straightforward for values within a small subset of the commitment group (e.g. 64-bit values within a 256-bit elliptic curve group), because of the wrap-around effect it was not clear how to do this efficiently for arbitrary values until our work. Further, we extend our approach to values that are much larger than the group order, using prime-order decomposition. The argument is efficient in that proof size is logarithmic in the size of the inputs and may be of independent interest.

Practical evaluation. We evaluate a case study of Ethereum Staking and find that our scheme can reduce bandwidth by up to a 20 \times factor compared to unweighted schemes. We also provide a proof-of-concept implementation and confirm that proof sizes are logarithmic, under 2 KiB for Ethereum. Running times scale linearly with the number of parties and the size of the secret, but with large constants. With a more optimized arithmetic circuit proof, our technique immediately becomes practical.

II. OVERVIEW

In this section, we give an overview of our methods and the required background while avoiding technical details. Please see Sections III and IV for full details.

A. CRT-based Secret Sharing

Let us begin with a high-level description of Chinese Remainder Theorem-based secret sharing [1], [12]. We assume a trusted dealer is given a secret $s_0 \in \mathbb{F}$, where \mathbb{F} is a field of size p_0 . To share the secret among n parties, we first choose n numbers, p_1, \dots, p_n , all co-prime to each other and p_0 . The choice of p_i values depends on the desired properties of the secret sharing scheme, which we discuss later. For a parameter L , the dealer chooses uniformly random $a \xleftarrow{\$} [L]$ and sets:

$$s = s_0 + ap_0 \leq (L + 1)p_0$$

We call s the lifted secret. The dealer then calculates the share for party i as $s_i = s \pmod{p_i}$, and sends each party

¹Ramp secret sharing has two threshold parameters: t and T . Parties with aggregate weight above T can reconstruct the correct secret, while parties with aggregate weight below t learn no information about the secret.

their share privately. Consider some set of parties $A \subseteq [n]$. Let $P_A = \prod_{i \in A} p_i$. By the Chinese Remainder Theorem, the system of equations $\{s_i = s' \pmod{p_i}\}_{i \in A}$ has a unique solution modulo P_A , which can be solved to obtain s' . If $P_A > (L + 1)p_0$, then $s' = s$, and we can find $s_0 = s' \pmod{p_0}$. Let \mathcal{A} be the set of all access sets authorized to reconstruct the secret. For reconstruction, we require that:

$$(L + 1)p_0 \leq \min_{A \in \mathcal{A}} P_A = P_{MIN}$$

Let $\bar{\mathcal{A}}$ be the set of all *unauthorized* access sets, define:

$$P_{MAX} = \max_{\bar{A} \in \bar{\mathcal{A}}} \prod_{i \in \bar{A}} p_i$$

It can be shown that the statistical distance between the solution to the CRT equations and uniformly random is at most $P_{\bar{A}}/L$ (see [9]), thus the security error of our scheme is upper bounded by P_{MAX}/L . Our desired property for both reconstruction and privacy of the system can thus be succinctly stated as:

$$P_{MAX} \ll L < P_{MIN}/p_0 - 1$$

Extension to Weighted Ramp Secret Sharing. The construction of a weighted secret-sharing scheme from the CRT construction of the previous subsection is brilliant in its simplicity. Let us assume that each party has an associated weight w_i , and that there are two thresholds: a reconstruction threshold T and a privacy threshold t . A set of parties A is authorized if $\sum_{i \in A} w_i \geq T$, and unauthorized if $\sum_{i \in A} w_i < t$. Notice that weights in between these two thresholds are neither authorized nor unauthorized. We pick each prime p_i such that it has bit length w_i , i.e. p_i is less than but close to 2^{w_i} . As a result, $P_{MAX} < 2^t$ and $P_{MIN} > 2^{T-O(1)}$. Let λ be our security parameter, and set p_0 to have bit length λ . If we set $L = 2^{\lambda+t}$, then we achieve security error $2^{-\lambda}$. Our system is correct if $T - t > 2\lambda + O(1)$. If the gap is insufficient, we can amplify all of the weights by a constant c . See [9] for more a more detailed analysis.

B. CRT-based Verifiable Secret Sharing

In addition to providing secret shares, in a VSS the dealer must also prove to the participants that the deal was correctly executed (see E for a formal definition). For CRT-based secret sharing schemes, the obvious approach is to provide commitments to the secret and the shares, and then prove in zero-knowledge that the committed share values are congruent to the original secret modulo the relevant prime. The participants verify the proofs, and that their share is consistent with the given commitments. If any of these checks fail they reject the deal. Otherwise, they use CRT reconstruction to obtain the secret when needed.

The main challenge is finding an appropriate commitment scheme and proof technique. Using group-based commitments (e.g. Pedersen's) the naïve approach leverages the homomorphic property: the prover provides a commitment $K = \text{Com}(k_i; r'_i)$ such that $s = s_i + k_i p_i$, and then the verifier checks that $S = S_i \oplus (p_i \otimes K_i)$, where \oplus is the commitment group operation, and \otimes is the scalar operation. This technique works if s is in a small subset of

the message space, but for arbitrary values, any group-based commitment scheme faces a wraparound issue. Specifically, if the commitment group has order p_0 , then the equation above is not over the integers but rather \mathbb{Z}_{p_0} . The dealer can always find k' such that $s = v + k'p \pmod{p_0}$ for any values v, s since $\gcd(p, p_0) = 1$. As proven in Appendix B, this problem affects all fixed-size homomorphic commitments.

Unstructured commitment schemes (such as hash-based schemes) avoid this problem, but there are currently no known efficient arithmetic circuit proofs for such schemes. Thus, for arbitrary field values, it is not clear how to efficiently construct a secure NIZK proof.²

C. Proof-of-Mod

The fundamental problem in creating a CRT-based VSS is proving that committed values satisfy $v = s \pmod{p}$ for a known prime p without revealing v or s . Here we give an overview of our "proof-of-mod" (PoM) protocol which achieves this goal, see Section III for full details.

a) Base PoM: To begin, suppose $s < p_0$ (i.e. s is within the order of the commitment group), and let $s = v + kp$, with $p_0 = qp + t$ where $0 \leq t < p$ (i.e. q is the quotient of p_0/p and t is the remainder). In order to prove that $v = s \pmod{p}$, it suffices to show (1) that $0 \leq v < p$ and (2) $0 \leq k < q$. The only other possibility is the last t values that s can take, in which case we instead need to show (1) $0 \leq v < t$ and (2) $0 \leq k \leq q$. Our key observation is that the PoM boils down to a disjunction of two sets of range proofs, each of which can be done within the group order:

$$\begin{aligned} & [(0 \leq v < p) \wedge (0 \leq k < q)] \\ & \vee [(0 \leq v < t) \wedge (0 \leq k \leq q)] \end{aligned} \quad (1)$$

b) Disjunction of Range Proofs: To understand how to prove a disjunction of ranges, we first recap the range-proof technique from Bulletproofs [5]. To prove that $0 \leq v_i < 2^n$, where v_i is the committed value, the prover shows that it knows the n -bit binary decomposition of v_i , by showing that it knows values a_j, b_j that satisfy the following equations (see Appendix F for a proof):

$$\sum_{j=1}^n a_{i,j} \cdot 2^{j-1} = v_i \quad (2)$$

$$a_{i,j} \cdot b_{i,j} = 0 \quad 1 \leq j \leq n \quad (3)$$

$$a_{i,j} - b_{i,j} - 1 = c_{i,j} = 0 \quad 1 \leq j \leq n \quad (4)$$

Notice that if the range holds, then (4) is always zero. If the range does *not* hold, then the prover can always find a_j, b_j that satisfy (2) and (3) but not (4). Thus, to prove a disjunction of $0 < v_1 < 2^n$ or $0 < v_2 < 2^n$, we can form a polynomial for each v_i that evaluates to zero if $\forall j, c_{i,j} = 0$, and non-zero with high probability otherwise. Let z be a random value chosen by the verifier. This produces the following equations:

$$\sum_{j=1}^n c_{1,j} \cdot z^{j-1} = c'_1, \quad \sum_{j=1}^n c_{2,j} \cdot z^{j-1} = c'_2 \quad (5)$$

²One could always use Succinct Non-interactive Arguments (SNARGs) for general circuits, but we believe a tailored NIZK proof would easily outperform the SNARK approach both in prover and verifier time.

If the disjunction is true then it must be the case that the product of the two is zero, i.e., $c'_1 \cdot c'_2 = 0$. If neither range holds, then with overwhelming probability the prover will not be able to fulfil the proof for a random challenge z . We refer the reader to Section III-A0b for more details.

c) Prime-order decomposition: For our CRT-based VSS to be useful, we require s to be larger than p_0 . To that end, we use the technique of "prime-order decomposition" which allows us to extend the base PoM above to arbitrarily large values. Consider the case that $s = s_0 + ap_0$ for some $a \in \mathbb{N}$, where $s_0 < p_0$. We now wish to prove that $v = s \pmod{p}$. Notice that we can now decompose the problem as follows:

$$\begin{aligned} v &= (s_0 + ap_0) \pmod{p} \\ &= (s_0 \pmod{p}) + (a \pmod{p})(p_0 \pmod{p}) \end{aligned}$$

If we assume that $p^2 + p < p_0$ then we can decompose the problem into three smaller proofs for relations: (1) ($s'_0 = s_0 \pmod{p}$), (2) ($a' = a \pmod{p}$), and (3) $v = s'_0 + a't \pmod{p}$, where $t = p_0 \pmod{p}$. The first two are simply applications of the base PoM, while the third is an arithmetic circuit followed by PoM. We can similarly extend this technique to any value polynomial in p_0 . For full details see Section III-B.

d) Efficiency: Proof sizes for the base PoM, extended PoM, and VSS are shown in Table I, assuming $p_0 < 2^\lambda$, n parties, and lifted secrets less than p_0^m . The base proof-of-mod is discussed in more detail in Section III, while the VSS proof is discussed in Section IV. The only additional communication in the protocol is the $n + 1$ commitments S_0, \dots, S_n , and $2n$ field elements $s_1, \dots, s_n, r_1, \dots, r_n$, as required to fulfill the definition of a VSS.

TABLE I: Proof sizes for PoM and VSS.

	Group Elements	Field Elements
Base PoM	$2\lceil \log_2(3\lambda + 1) \rceil + 8$	6
Ext. PoM	$2\lceil \log_2 3m(3\lambda + 1) \rceil + 8$	6
VSS	$2\lceil \log_2 3nm(3\lambda + 1) \rceil + 8$	6

D. Practical Parameters

To use a single prime order group, we make the assumption that the size of the group is large enough that all of the sharing primes p_i are smaller than the group order. As there is also a limit on how small the sharing primes can be in practice, this necessarily puts a limit on the dynamic range of weights that can be supported. If our group order is $p_0 < 2^N$, then as we will see in Section III-B, the maximum weight supported is less than $N_{max} \leq N/2$. On the other end, if the minimum practical weight is N_{min} , then the dynamic range available is $N/2 - N_{min}$.

For example, if we assume a 256-bit group order (such as curve Sec256k1), and a minimum prime length of 10 bits, then the dynamic range³ available is 118. In other words, if the largest party has 10% of the total weight, then the smallest party that can take part has weight 0.085%. We feel this is sufficient for most practical purposes, including Ethereum (see Section V), however if a larger dynamic range is needed, then there are several solutions:

³Dynamic range is the ratio of the maximum to the minimum value.

- 1) *Use a larger order group*: For example, Curve448 has order approximately 446 bits. With the same lower limit of 10 bits, this produces a dynamic range of 213. Using our example above, if the largest party has 10% of the total weight, then the smallest party is now 0.047% of the total weight.
- 2) *Weight pooling*: Similar to virtualization, it may be possible at a protocol level to have very small weight participants pool their weight together if they agree to act as one party. While this goes against the design of the protocol at a high level, for very small participants, this may be an acceptable solution if it results in considerable efficiency improvements.
- 3) *Virtualization*: On the opposite extreme, if there are very few very large participants, it may be possible to virtualize their shares. For example, if one party has 33% of the total weight, then they could be split into three virtual parties, each with 11% of the total weight. If all other parties have weight less than 11%, then only this party incurs the extra overhead. This produces significantly better fairness for smaller parties, while keeping the overhead for the large party quasi-logarithmic.

E. Publicly Verifiable Secret Sharing

Notice that the above scheme requires a round of interaction between the participants to ensure a quorum of sufficient size has received valid shares and accepts the deal. As described in [16], we can eliminate this round of interaction to implement what is called *Publicly Verifiable Secret Sharing (PVSS)* if we use a Verifiable Encryption scheme, such as Verifiable ElGamal (sometimes called “El-Gamal in the exponent”). This allows the dealer to broadcast encryptions of the share values, where the share of the i -th party is verifiably encrypted under the i -th party’s public key. The dealer then attaches a NIZK proof to prove to (all) the participants that the encrypted values match the committed values. Thus the participants can verify that the shares of all parties are correct without revealing them and without any interaction. Our construction is amenable to the PVSS scheme of [16] to eliminate interaction between participants at the cost of a larger broadcast to all participants.

F. Notation and Definitions

We use λ for the security parameter, which may be implicit in a set of parameters, for example the parameters of the verifiable secret sharing scheme. We use \mathbb{F} to denote an arbitrary field, and \mathbb{Z}_p to denote the field of integers modulo a prime p . Let $\text{negl}(\lambda)$ denote a negligible function, that is for all polynomial $p(\lambda)$, there exists λ_0 such that $\text{negl}(\lambda) < 1/p(\lambda) \forall \lambda > \lambda_0$. We denote by $[n]$ the set $\{1, \dots, n\}$. We use the \vec{v} notation for vectors and uppercase bold font \mathbf{M} for matrices.

a) *Commitments*: We use $\text{Com}(x; r) \rightarrow c$ for a commitment to x using randomness r . Once an adversary provides c knowing x and r , it should be hard to find another pair (x', r') such that $c = \text{Com}(x'; r')$ (binding). Without x and r , it should also be hard to determine which x was committed to (hiding). See Appendix A for details and formal definitions.

b) *Arithmetic Circuit Proofs*: We will make extensive use of Arithmetic Circuit Proofs, a special case of Zero Knowledge Proofs (ZKP)⁴. A ZKP convinces a verifier that the prover knows a value x satisfying a relation \mathcal{R} without revealing x . Here, the prover shows that committed values satisfy an arithmetic circuit, defined by multiplication gates $\vec{a} \circ \vec{b} = \vec{c}$ and constraints on input and output values⁵. We denote the prover and verifier algorithms as $\Pi_{CKT} = (P_{CKT}, V_{CKT})$, where $P_{CKT}((\vec{V}, \mathbf{CKT}, A, B, C); (\vec{v}, \vec{a}, \vec{b}, \vec{c}, \vec{r}_v, (r_a, r_b, r_c)))$ is the prover taking input commitment \vec{V} , circuit definition \mathbf{CKT} , wire commitments A, B, C , private inputs \vec{v} , wire values $\vec{a}, \vec{b}, \vec{c}$, and randomness \vec{r}_v, r_a, r_b, r_c . The verifier $V_{CKT}(\vec{V}, \mathbf{CKT}, A, B, C) \rightarrow \phi_{CKT}$ takes the same public inputs and returns $\phi_{CKT} \in \{\text{Accept}, \text{Reject}\}$. See Appendix C and D for formal definitions and further details, and [5] for an example implementation.

III. PROOF-OF-MOD

In this section we construct a NIZK proof that a committed value is the result of taking another committed value modulo a known prime. We start with the case that all values are within the order of the discrete log group, and then show how to extend this technique to arbitrary values. We build these proofs by constructing an arithmetic circuit that is satisfied only if the proof condition holds. Since such circuit proofs may be aggregated (see [5]), we can combine many such proofs into a much smaller proof than if they were constructed individually.

A. Initial Construction

a) *Range Proof Circuits*: As we discussed in Section II-C, we can use Equations 2 to 4 to construct an arithmetic circuit that is satisfied only if the input value v_i is in the range $[0, 2^n]$. We can tighten the range to $0 \leq v_i < u_i$ by repeating this subcircuit, replacing v_i with $u_i - v_i$. This subcircuit is easily written in the form needed by circuit proofs (see Appendix D), requires $2n$ multiplications, and $2n + 2$ constraints.

b) *Disjunction of Ranges*: We now construct a circuit for a disjunction of ranges, that is, *either* $0 \leq v_1 < 2^n$ *or* $0 \leq v_2 < 2^n$. Let us assume that the prover chooses $a_{i,j}$ and $b_{i,j}$ such that (2) and (3) hold for both values, but (4) holds for one of v_1, v_2 but not the other. In this case, we wish to prove that either *all*. $c_{1,j}$ values are zero, or *all* $c_{2,j}$ values are zero. To this end we construct two polynomials:

$$\sum_{j=1}^n c_{i,j} \cdot z^{j-1} = c'_i \quad i \in \{1, 2\} \quad (6)$$

After the prover has committed to the $a_{i,j}$ and $b_{i,j}$ values, the verifier chooses a random $z \in \mathbb{F}$. If all of the $c_{i,j}$ values are zero then c'_i will be zero for any choice of z . If at least one of the $c_{i,j}$ values is non-zero, then c'_i will be non-zero with overwhelming probability. We can add one more multiplication gate for $c'_1 \cdot c'_2 = 0$. If one of the ranges

⁴A ZKP for computationally bounded adversaries is technically an Argument, but we use the terms interchangeably.

⁵Arithmetic circuits of this form are also called Rank 1 Constraint Systems (R1CS).

hold, then this gate is satisfied. If neither range holds, then with very high probability $c'_1 \cdot c'_2 \neq 0$. Notice that we can replace the n constraints in equation (4) with the single constraint in equation (6) for each element in the disjunction by eliminating the intermediate variables, thus reducing the number of constraints from $2n$ to 2.

It may seem that the prover cannot satisfy the circuit without first knowing the choice of z , but there is a solution. The prover sets $a_{i,1} = v_i$, $b_{i,1} = 0$, and all other $a_{i,j} = 0$, $b_{i,j} = -1$. As a result, $c_{i,1} = v_i$ and all other $c_{i,j} = 0$ satisfies the circuit for any value z chosen by the verifier.

c) *Base Proof-of-Mod*: As described in Section II-C, we first construct a ZKA for a congruence relationship between two values when all values are less than the order of the group used for the commitments, which we call “Base Proof-of-Mod”.

Definition 1 (Base Proof-of-Mod). *Given a commitment scheme (Setup, Com) over a message space $\mathcal{M} \subset \mathbb{N}$, a proof-of-mod is a zero-knowledge argument of knowledge for the relation \mathcal{R}_{mod} :*

$$\mathcal{R}_{\text{mod}}((V, S, p), (v, s, r_v, r_s)) := \\ V = \text{Com}(v, r_v) \wedge S = \text{Com}(s, r_s) \wedge v = s \pmod{p} \quad (7)$$

Where $V, S \in \mathcal{C}$, $p \in \mathbb{N}$, $v, s \in \mathcal{M}$, and $r_v, r_s \in \mathcal{R}$, and the modulo operation is over the natural numbers.

As discussed in Section II-C, the prover must show that there exists a value k such that $s = v + kp$ and that the resulting value does not wrap around, i.e. that $v + kp < p_0$. This results in the disjunction of ranges shown in Equation 1, which we can use to construct a circuit that is satisfied if and only if the proof condition holds. Let $p_0 = |\mathcal{M}|$ be the order of the commitment scheme, which we assume to be prime, and $p_0 = qp + t$ where $q, t \in \mathbb{N}$ and $0 \leq t < p$. Our circuit takes as input v, s and is constructed as follows:

- 1) Write constraint $s = v + kp$ for intermediate variable k .
- 2) Write range subcircuits for $0 \leq v < p$ and $0 \leq k \leq q$.
- 3) Write disjunction subcircuit for $k < q$ or $v < t$.

We show an optimized version of this circuit in Appendix G, which uses $3n_1 + 3n_2 + 1$ multiplications and $5n_1 + 5n_2 + 9$ constraints, where $p < 2^{n_1}$ and $q < 2^{n_2}$. Let $\text{ModCkt}(p_0, p, z) \rightarrow \mathbf{CKT}$ be a function that generates the circuit specification using the supplied constants.

Let $\text{ModSolve}(v, s, p_0, p) \rightarrow \vec{a}, \vec{b}, \vec{c}$ be a function that produces the wire values that solve the circuit produced by $\text{ModCkt}(p_0, p, z)$ for the given values. An implementation of ModSolve is shown in Appendix G. We can now write the base proof-of-mod as shown in Figure 1.

1) *Security*: Security rests on the security of the commitments and the circuit proofs. We state our security theorem below, and provide a proof.

Theorem 1. *If the commitment scheme (Setup, Com) used is computationally binding and perfectly hiding, and the circuit proof protocol Π_{CKT} is perfectly complete, provides computational soundness, and special honest verifier zero-knowledge, then Π_{MOD} is perfectly complete, provides*

computational soundness, and special honest verifier zero-knowledge.

Proof. Completeness. Given v, s, r_v, r_s such that $v = s \pmod{p}$, the honest prover can always find a value of k and all intermediate values that satisfies the circuit for any value of z . From the completeness of the circuit proof protocol, the verifier will always accept.

Computational Soundness. Consider a cheating prover P^* that is given v, s, r_v, r_s such that $v \neq s \pmod{p}$, and both the verifier and prover are given V, S, p . Assume $S = \text{Com}(s, r_s), V = \text{Com}(v, r_v)$, and $0 \leq v < p$ since this is the most advantageous condition for the prover. Consider the following cases.

Case 1. P^* finds either v', r'_v such that $V = \text{Com}(v', r'_v)$ and $v' = s \pmod{p}$, or s', r'_s such that $S = \text{Com}(s', r'_s)$ and $v = s' \pmod{p}$. In either case the prover has broken the binding property of the commitment scheme, which by assumption the probability of any PPT adversary finding such values is negligible.

Case 2. P^* sends commitments $A \leftarrow \text{Com}(\vec{a}; r_a), B \leftarrow \text{Com}(\vec{b}; r_b), C \leftarrow \text{Com}(\vec{c}; r_c)$, but then finds another set of openings $\vec{a}', \vec{b}', \vec{c}', r'_a, r'_b, r'_c$ such that $A = \text{Com}(\vec{a}'; r'_a), B = \text{Com}(\vec{b}'; r'_b), C = \text{Com}(\vec{c}'; r'_c)$ that satisfy the circuit. In this case, the prover has broken the binding property of the commitment scheme. By assumption the probability of any PPT adversary finding such values is negligible.

Case 3. P^* sends commitments A, B, C to values $\vec{a}, \vec{b}, \vec{c}$ that do not satisfy the circuit, yet the prover convinces the verifier to accept. In this case, the prover has broken the soundness of Π_{CKT} . By assumption the probability of any PPT adversary finding such values is negligible.

Case 4. P^* finds values that satisfy the disjunction section of the circuit despite the disjunction being false. More precisely, at least one of $a_{5,1} - b_{5,1} - 1 \neq 1$ and at least one of $a_{6,1} - b_{6,1} - 1 \neq 1$, yet one of c'_1, c'_2 is zero (see Equation 6 in Section III-A0b). By the fundamental theorem of algebra, a polynomial of degree n has at most n roots. Thus, if z is chosen uniformly at random, after the polynomial is fixed, then the probability of this occurring is $2n/|\mathbb{F}|$. Assuming n grows at most linearly with the security parameter λ and $|\mathbb{F}|$ is exponential in the same, then the probability of this occurring is negligible.

Case 5. If none of the above cases occur, then the ranges specified in Section II-C are satisfied, and there exists $k \in \mathbb{F}$ such that $s = v + kp < p_0$.

Since the three cases above cover all possibilities, from the law of total probability it can be seen that the advantage of any PPT adversary in the soundness game is negligible.

Special Honest Verifier Zero-Knowledge. We show SHVZK by building a simulator \mathcal{S} that uses the simulator S_{CKT} of Π_{CKT} to simulate the protocol. First, \mathcal{S} chooses $z \in \mathbb{F}$ uniformly at random, and constructs the circuit using $\text{ModCkt}(p_0, p, z)$. It then calls S_{CKT} to get the transcript of the circuit proof including the commits A, B, C . It then outputs the transcript with the addition of z after A, B, C are issued by the prover. By the SHVZK property of Π_{CKT} , the transcripts produced by \mathcal{S} must have the same

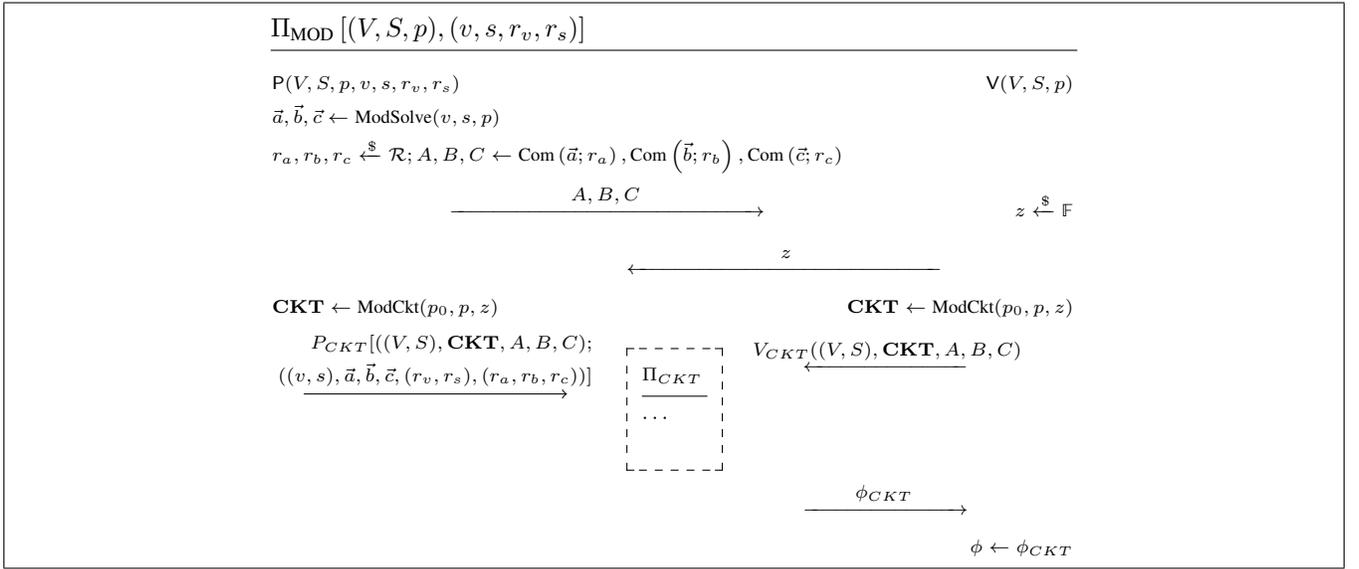


Fig. 1: Base Proof-of-Mod Protocol.

distribution as the transcripts produced by the honest run of the protocol. \square

2) *Efficiency*: The circuit proof protocol Π_{CKT} given in [5] requires $2\lceil \log_2 n \rceil + 8$ group elements and 5 field elements, where n in this case is the number of multiplication gates. Our protocol adds only one field element, and uses $3\lambda + 1$ multiplications, where λ is the bit length of p_0 . Thus, in total our protocol requires $2\lceil \log_2(3\lambda + 1) \rceil + 8$ group elements and 6 field elements.

B. Prime-Order Decomposition

As discussed in Section II-C, we can extend the base proof-of-mod to arbitrary values by a technique we call “prime-order decomposition”, in which we decompose s using p_0 as a base. Consider that we can always write s as:

$$s = a_0 + a_1 p_0 + a_2 p_0^2 + \dots + a_m p_0^m = \sum_{i=0}^m a_i p_0^i$$

Where $a_i \in \mathbb{Z}_{p_0} \forall i \in [m]$. We can now define an extended version of the proof-of-mod as follows.

Definition 2 (Extended Proof-of-Mod). *Given a commitment scheme (Setup, Com) over a message space $\mathcal{M} \subset \mathbb{N}$, a proof-of-mod is a zero-knowledge argument of knowledge for the relation \mathcal{R}_{EMOD} :*

$$\begin{aligned} \mathcal{R}_{EMOD}((V, A_0, \dots, A_m, p), \\ (v, r_v, a_0, \dots, a_m, r_0, \dots, r_m)) := \\ v = \sum_{i=0}^m a_i p_0^i \bmod p \wedge V = \text{Com}(v, r_v) \wedge \\ A_i = \text{Com}(a_i, r_i) \forall i \in [m] \end{aligned}$$

Where $V, A_0, \dots, A_m \in \mathcal{C}$, $p \in \mathbb{N}$, $v, a_0, \dots, a_m \in \mathcal{M}$, and $r_v, r_0, \dots, r_m \in \mathcal{R}$, and the modulo operation is over the natural numbers.

We will again take a circuit-building approach in order to prove \mathcal{R}_{EMOD} . First, notice that we can decompose the problem using Horner’s method:

$$v = a_0 + p_0(a_1 + p_0(a_2 + \dots + (a_{m-1} + p_0 a_m))) \bmod p \quad (8)$$

Let us define the intermediate steps in (8) as $v_i = a_i + p_0 v_{i+1} = a_i + p_0(\dots + p_0 a_m)$, where $v_m = a_m$. Assuming $p_0 > p^2 + p$, then we can ensure that all values are within \mathbb{Z}_{p_0} by taking each step modulo p as follows. Let $a'_i = a_i \bmod p$ and $t = p_0 \bmod p$. Define:

$$\begin{aligned} v'_i &= v_i \bmod p \\ &= a_i + p_0 \cdot v_{i+1} \bmod p \\ &= a'_i + t \cdot v'_{i+1} \bmod p \quad \forall 0 \leq i < m \end{aligned}$$

We can now apply the proof-of-mod circuit from Section III-A first to the a'_i values, and then to each of the v'_i , to build a circuit that proves the correctness of v with respect to a_0, \dots, a_m . More concretely, we construct the circuit as follows:

- 1) Write proof-of-mod subcircuits for $a'_i = a_i \bmod p$ for $i \in [0, m]$.
- 2) Let $v'_m = a'_m$. For j in $[0, m - 1]$ write:
 - a) Constraint $v'_j = a'_j + t \cdot v'_{j+1}$.
 - b) Proof-of-mod subcircuit for $v'_j = v_j \bmod p$.
- 3) Write constraint $v = v'_0$.

In practice, the a'_i, v_i, v'_i variables would be eliminated and instead the bit decompositions used directly, but we present this method for ease of understanding. If $p_0 > mp^2$ then instead of m PoM subcircuits for the v'_i steps, we can directly write a constraint for v from the a'_i values. Let $\text{EModCkt}(p_0, p, z) \rightarrow \text{CKT}$ be a function that generates the circuit specification using the method described above, and let $\text{EModSolve}(v, a_0, \dots, a_m, p_0, p)$ be a function that produces the wire values that solve CKT . The resulting protocol is identical to the previous protocol, replacing ModSolve with EModSolve and ModCkt with EModCkt . Thus we elide the protocol diagram and proof here. We

denote the extended proof-of-mod protocol proving \mathcal{R}_{EMOD} as Π_{EMOD} .

Efficiency. By constructing the proof as one large circuit we achieve considerable efficiency in terms of proof size. As in the previous subsection, each PoM requires $3\lambda + 1$ multiplication gates, where $p_0 < 2^\lambda$. The full construction uses $2m$ such PoM circuits and no additional multiplications. Thus, the total communication cost is $2\lceil \log_2 m + \log_2(3\lambda + 1) + \log_2 3 \rceil + 8$ group elements and 6 field elements.

a) Non-Interactive Construction: The Fiat-Shamir heuristic [8] allows us to convert any public-coin interactive protocol into a non-interactive one with computational soundness by replacing any randomness used by the verifier with a hash of the transcript to that point. As is discussed in detail in [5], the circuit proof protocol Π_{CKT} can thus be rewritten in the form of a prover procedure that produces a transcript, π_{ckt} , and a verifier procedure that takes π_{ckt} as input and outputs accept or reject.

Similarly, we can convert our proof-of-mod protocol into a non-interactive proof simply by setting the verifier's challenge, z to be the hash of the commitments A, B, C and the inputs to the protocol. We then use the non-interactive version of the circuit proof as a subroutine to get the full proof. We show the full details of the non-interactive proof-of-mod protocol in Appendix 15.

IV. CRT-BASED VSS

Armed with the extended proof-of-mod we constructed in Section III, along with appropriate commitment and arithmetic circuit proof schemes, we are now ready to flesh out the CRT-based VSS scheme for which we gave a high level description in Section II-B. We first construct a proof that *all* of the shares are correct by grouping all of the proof-of-mods into a single circuit, and treating the a_0, \dots, a_m values as intermediate variables that hold for all of the secret shares. This way, separate commitments A_1, \dots, A_m are not necessary. We can construct such a circuit taking s_0, \dots, s_n as inputs as follows:

- 1) Let a_0, \dots, a_m be intermediate variables
- 2) Write constraint that $a_0 = s_0$.
- 3) For i in 1 to n , write an extended proof-of-mod circuit for $s_i = \sum_{j=0}^m a_j p_0^j \pmod{p_i}$.

As in the Extended Proof-of-Mod construction, the intermediate variables will be eliminated in an optimized circuit, but their existence and consistency is still guaranteed. Let $\text{VSSCKT}(p_0, \dots, p_n, z) \rightarrow \text{CKT}$ be a function that constructs the circuit as described above using z as the challenge value for the proof-of-mod subcircuits, and let $\text{VSSSolve}(s_0, \dots, s_n, a_1, \dots, a_m, p_0, \dots, p_n) \rightarrow \vec{a}, \vec{b}, \vec{c}$ be a function that produces the wire values that solve **CKT** for any value of z . We can now write our non-interactive VSS construction by defining the Share and Reconstruct programs as shown in Algorithms 1 and 2 respectively.

This construction uses n extended proof-of-mod circuits, each of which uses $3m(2\lambda + 1)$ multiplication gates. Thus, the size of π_{ckt} is $2\lceil \log_2 n + \log_2 m + \log_2(3\lambda + 1) + \log_2 3 \rceil + 8$ group elements and 6 field elements, including the commitments A, B, C . The only additional outputs are the

$n + 1$ commitments Y_0, \dots, Y_n in π , and $2n$ field elements $s_1, \dots, s_n, r_1, \dots, r_n$.

Algorithm 1: VSS Share Procedure

Input: Secret $s_0 \in \mathbb{Z}_{p_0}$, primes p_0, \dots, p_n
Output: Shares v_1, \dots, v_n , proof π

- 1 $a_1, \dots, a_m \xleftarrow{\$} \mathbb{Z}_{p_0}; \vec{r} = r_1, \dots, r_n \xleftarrow{\$} \mathcal{R};$
 $Y_0 \leftarrow \text{Com}(s_0; r_0);$
- 2 **for** $i = 1$ **to** n **do**
- 3 $s_i \leftarrow s_0 + \sum_{j=0}^m a_j p_0^j \pmod{p_i};$
- 4 $Y_i \leftarrow \text{Com}(s_i; r_i);$
- 5 **end**
- 6 $\vec{a}, \vec{b}, \vec{c} \leftarrow$
 $\text{VSSSolve}(s_0, \dots, s_n, a_1, \dots, a_m, p_0, \dots, p_n);$
- 7 $r_a, r_b, r_c \xleftarrow{\$} \mathcal{R}; A, B, C \leftarrow$
 $\text{Com}(\vec{a}; r_a), \text{Com}(\vec{b}; r_b), \text{Com}(\vec{c}; r_c);$
- 8 $z \leftarrow H(p_0, \dots, p_n, Y_0, \dots, Y_n, A, B, C);$
- 9 **CKT** $\leftarrow \text{VSSCKT}(p_0, \dots, p_n, z);$
- 10 $\pi_{ckt} \leftarrow \text{tr} < \Pi_{CKT} > ((Y_0, \dots, Y_n, A, B, C),$
 $\text{CKT}, (s_0, \dots, s_n), \vec{a}, \vec{b}, \vec{c}, \vec{r}, r_a, r_b, r_c);$
- 11 $\pi \leftarrow (Y_0, \dots, Y_n, A, B, C, \pi_{ckt});$
- 12 **return** $(s_0, r_0), \dots, (s_n, r_n), \pi;$

Algorithm 2: VSS Reconstruct Procedure

Input: Primes p_0, \dots, p_n , shares $\{(i, v_i)\}_{i \in A}$, proof π
Output: Secret s' or \perp

- 1 $Y_0, \dots, Y_n, A, B, C, \pi_{ckt} \leftarrow \pi; s_i, r_i \leftarrow v_i \forall i \in A;$
- 2 $z \leftarrow H(p_0, \dots, p_n, Y_0, \dots, Y_n, A, B, C);$
- 3 **CKT** $\leftarrow \text{VSSCKT}(p_0, \dots, p_n, z);$
- 4 **if**
 $V_{CKT}((Y_0, \dots, Y_n), \text{CKT}, A, B, C, \pi_{ckt}) \neq \text{Accept}$
 $\text{or } \exists i \in A | Y_i \neq \text{Com}(s_i, r_i)$ **then**
- 5 \mid **return** $\perp;$
- 6 **end**
- 7 $P_A \leftarrow \prod_{i \in A} p_i;$
- 8 $q_i \leftarrow \prod_{j \in A \setminus \{i\}} p_j; q'_i \leftarrow q_i^{-1} \pmod{p_i} \forall i \in A;$
- 9 $s' \leftarrow \sum_{i \in A} s_i q_i q'_i \pmod{P_A};$
- 10 **return** $s' \pmod{p_0};$

a) Security: The security of our scheme rests on the security of the building blocks. Secrecy follows from the hiding property of the commitment scheme and the SHVZK property of the circuit proof. Correctness follows from the perfect correctness of the PoM circuit. Finally, commitment follows from the binding property of the commitment scheme, and the soundness of the circuit proof. We state our final theorem below and provide a proof.

Theorem 2. *If the commitment scheme is computationally binding and perfectly hiding, and the circuit proof used provides perfect completeness, computational soundness, and SHVZK, then the NI-VSS shown in Algorithms 1 and 2 is secure using Definition 9.*

Proof. Secrecy. Consider an arbitrary unauthorized access set $\bar{A} \in \bar{\mathcal{A}}$, and any two secrets s, s' . By the hiding

property of the commitment scheme, the distribution of Y_0, \dots, Y_n, A, B, C are independent and indistinguishable for any two secrets s, s' . Similarly, by the SHVZK property of the circuit proof, π_{ckt} is indistinguishable for any two secrets s, s' . The choices of $\{r_i\}_{i \in \bar{A}}$ are independent and uniformly random, and thus also indistinguishable. Finally, we assume that p_0, \dots, p_n are chosen such that for any $\bar{A} \in \bar{\mathcal{A}}$, $P_{\bar{A}} \leq P_{max} < p_0^{m+1}$. From [9], we can state the the statistical distance of $\{s_i\}_{i \in \bar{A}}$ from uniform is at most P_{max}/p_0^{m+1} . Thus, if $P_{max} \ll p_0^m$, then the statistical distance between the output of Share_{pp} for any s, s' is negligible.

Correctness. Consider an arbitrary $A \in \mathcal{A}$. First, note that the output of Share_{pp} will always be such that $s_i = s_0 + \sum_{i=1}^{m-1} a_i p_0^m \bmod p_i$, $Y_i = \text{Com}(s_i; r_i) \forall i \in [n]$. Next, from the perfect correctness of the extended proof-of-mod circuit and the circuit argument, the output values A, B, C, π_{ckt} are such that if $z = H(A, B, C)$ and $\mathbf{CKT} = \text{VSSCkt}(p_0, \dots, p_n, z)$, then $V_{CKT}((Y_0, \dots, Y_n), \mathbf{CKT}, A, B, C, \pi_{ckt})$ will always accept. Finally, by the Chinese Remainder Theorem, if p_0, \dots, p_n are chosen such that $P_A > P_{min} > p_0^{m+1}$, then the output of Reconstruct_{pp} will always be $s' = s_0$.

Commitment. To prove that no adversary D^* can in time polynomial in the security parameter produce a proof and two sets of shares $\pi, \{v_i\}_{i \in A}, \{v'_i\}_{i \in A'}$ where $A, A' \in \mathcal{A}$ such that the two sets of shares reconstruct different secrets, we consider the following cases.

Case 1. Assume in this case that D^* is able to commit to shares s_1, \dots, s_n as Y_1, \dots, Y_n that are consistent with the honest protocol, but then provides an opening for at least one of the shares for a different value, $Y_i = \text{Com}(s_i; r_i) = \text{Com}(s'_i; r'_i)$. By the computational binding of the commitment scheme, the probability of a PPT D^* finding such values is negligible.

Case 2. In this case, assume the dealer commits to s_1, \dots, s_n as Y_1, \dots, Y_n such that the arithmetic circuit is not satisfied, yet manages to produce a transcript π_{ckt} such that $V_{CKT}((Y_0, \dots, Y_n), \mathbf{CKT}, A, B, C, \pi_{ckt})$ accepts. By the computational soundness of the circuit proof, the probability of any PPT D^* producing such a proof is negligible.

Case 3. In this case, assume that the commitments are binding and that the circuit proof is sound, but that D^* is able to guess values of $\vec{a}, \vec{b}, \vec{c}$ that satisfy the circuit for the value of z obtained by the hash function. By the random oracle model, we can model H as a uniformly random function. Thus the probability of z satisfying the circuit for a given choice of $\vec{a}, \vec{b}, \vec{c}$ is at most $n/|\mathbb{F}|$ from the same argument given in Section III-A1. Thus, if n grows at most linearly in the security parameter, and $|\mathbb{F}|$ grows exponentially, then each attempt has negligible probability of success, and a polynomial number of such guesses by D^* still results in negligible probability of success.

Case 4. In this case, we assume that the commitments are binding, the circuit proof is sound, and the choice of z is such that no false roots are found. It must be the case that all of the equations in the circuit are satisfied. Thus, there exists $a_0, \dots, a_m \in \mathbb{Z}_{p_0}$ such that for all $i \in [n]$, $s_i = \sum_{j=0}^m a_j p_0^j \bmod p_i$. As a result, the reconstruction program will always

output $s' = a_0$.

By the law of total probability, the probability of D^* producing inconsistent share values that pass the verification program is negligible. \square

b) Practical Parameters: As we discussed in Section II-A, in the WRSS scheme, the weight of participant P_i is determined by the length of their associated prime p_i . In our construction, $p_0 > p_i^2 + p_i$, and if $p_0 < 2^N$, then $p_i < 2^{\frac{N-1}{2}}$. If the minimum practical value of p_i is N_{min} , then the dynamic range supported by the scheme is $N/2N_{min}$. For example, in Curve25519, p_0 is approximately 2^{255} , thus if $N_{min} = 10$ then $2^{10} < p_i < 2^{127}$ and our dynamic range is 12.7. Beyond this range parties must again be virtualized, but notice that the virtualized parties themselves have weight. Thus, the virtualization process is not only more efficient, in that the number of virtual parties is reduced by up to an order of magnitude, but also more fair, as the weights are cumulative.

V. PERFORMANCE

To evaluate the performance of our VSS, we take a case study of Ethereum staking. We then provide a theoretical analysis of the overhead of using a traditional VSS scheme with virtualization as compared to our scheme. Finally, we implement our scheme to see the operating parameters in practice.

a) Ethereum Staking: Staking is the process used by proof-of-stake blockchains to validate transactions. To participate, a user must deposit a certain amount of Ether (the native currency of Ethereum) into a smart contract, which is called their ‘‘stake’’. The user is then allowed to participate in the consensus mechanism, and is rewarded for doing so. If the user misbehaves, their stake is slashed, i.e. a portion of their deposit is destroyed. A minimum stake of 32 Ether is required to participate. [15]

The current system uses aggregate BLS signatures, which according to one analysis [4], processes 28,000 signatures per slot, and may rise to 1.8 million. Even with extremely efficient aggregate signatures, this is a large overhead. The resulting distribution of stakes is extremely wide, with the top two staking pools (Lido and Coinbase) controlling 45% of the total stake. We show the distribution of the rest of the system in Figure 2.

b) Comparing Virtualization to WR-VSS: Suppose Ethereum adopts a consensus scheme based on threshold signatures, wherein validators receive shares of the signing key and collaborate to jointly sign proposed blocks. This would require a full distributed key generation protocol, but for the sake of comparison consider only VSS overhead. As a baseline, consider the Feldman VSS [7], a simple yet efficient VSS that uses Shamir’s Secret Sharing scheme applied to discrete-log groups⁶. To share a secret among N (virtual) parties with threshold t , the scheme requires $N + t$ group elements to be broadcast, and one field element sent privately to each (virtual) party.

⁶More recent work provides better security, but to our knowledge, it does not use less bandwidth.



Fig. 2: Distribution of Ethereum Stakes for pools other than Lido and Coinbase. Note that the x-axis is logarithmic.

From our analysis, such a scheme would require 4,110 virtual parties, and if $t = 2N/3$ then 6,850 group elements must be broadcast and 4,110 field elements sent privately to each party. This assumes a minimum stake of 0.02% which excludes only 0.02% of the current staking. This is not a large improvement over the current design, and has problems with fairness⁷.

By contrast, if Ethereum used our Weighted-Ramp VSS to generate the signing key with a reconstruction threshold set to $2/3$ of the total weight, and if the minimum stake of 0.02% corresponds to a weight of 10, then the total weight in the system is 41,125. Using a curve of order $\sim 2^{255}$, we set $m = 108$ and need 365 parties. As can be seen in Table II, this produces more than a $100\times$ improvement on the current design, and nearly $20\times$ improvement in broadcast and $5\times$ improvement in private bandwidth compared to the virtualized VSS.

Byte sizes assume Curve25519 for both VSS schemes, and BLS381 (48 byte group elements) for the current design. One last benefit to the WRSS approach is that depending on the choice of signature scheme, it may not be necessary to use a pairing-friendly curve (e.g. BLS12-381 as is currently in use) and instead, use a simpler curve (e.g. Curve25519 or sec256k1), which both simplifies the overall scheme and reduces the bandwidth overhead.

TABLE II: Comparison of bandwidth usage for current Ethereum signature broadcast, Feldman-VSS, and our WR-VSS with 0.02% minimum stake and $t = 2/3N$.

Design	Broadcast			Private	
	G	Z_{p_0}	Total (B)	Z_{p_0}	Total (B)
Current	28,000		1,344,000		
Feldman	6,850		219,200	4,110	131,520
WR VSS	389	6	12,640	~ 892	28,528

c) Implementation: To evaluate our scheme’s practical performance, we implemented it in Rust using the Bulletproofs library from [6] for RICS proofs⁸. We tested on an Intel Xeon Gold 6230 (2.10 GHz) running Ubuntu 20.04.2 LTS. Prover running times for various configurations are

⁷E.g. if a validator has 0.05% of the total stake, should they receive 2 or 3 votes?

⁸RICS, short for Rank-1 Constraint System, is another term for arithmetic circuits.

shown in Figure 3. As expected, time is linear in both n (number of parties) and m (lifting value). However, the constants are large; even with this linear relationship, over 5 minutes is needed for just 4 parties with $m = 4$. We extrapolate from this that applying our scheme to Ethereum would require hours of prover time. Note that our implementation is not optimized, and the Bulletproofs library is still in development, so significant improvements may be possible.

The main advantage of our scheme is its small proof size. As shown in Figure 3, proof size is logarithmic in both m and n , with small constants. We extrapolate that for the Ethereum use cases, proof size should be under 2 KiB (excluding commitments), aligning with theoretical expectations.

Thus, we have shown that our scheme achieves its primary goal of proof size, though currently due to the poor performance of RICS libraries, the prover time is too high to be used in practice.

VI. CONCLUSIONS AND FUTURE WORK

In this work we have shown how to construct a Verifiable Weighted-Ramp Secret Sharing scheme. Our scheme is based on the CRT and is the first to provide verifiability for a weighted secret sharing scheme without a trusted setup, unknown order groups, or the strong RSA assumption. Along the way we developed a novel technique for proving congruence relationships between committed values in zero knowledge that may be of independent interest. We have shown that our scheme is efficient, with communication costs scaling logarithmically in all parameters, and running times linear in all parameters.

At present, the scheme is implemented using the Bulletproofs arithmetic circuit proof system, which introduces large constants in the prover time. Future work may optimize this by using more efficient arithmetic circuit proof systems, at which point the prover time may be more practical for large systems. It may also be possible to trade off some of the prover time for increased proof size. Finally, future work may consider the case that the a party’s prime value is greater than the group order, which would require a different proof-of-mod system, in order to enable larger dynamic ranges.

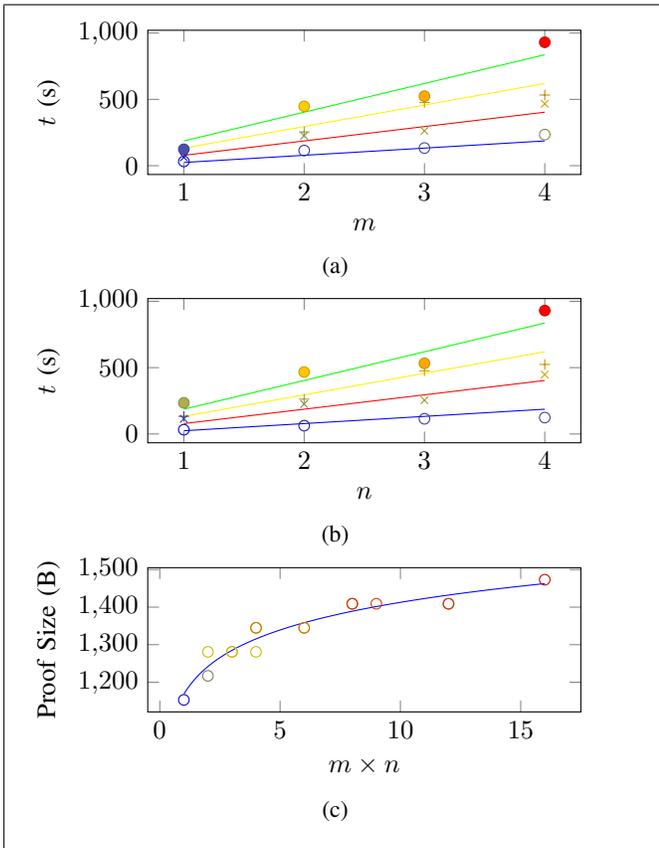


Fig. 3: VSS Measurements: (a) prover time vs m (size of the lifted secret), (b) prover time vs n (number of parties), (c) proof size.

REFERENCES

- [1] C. Asmuth and J. Bloom, "A modular approach to key safeguarding," *IEEE Transactions on Information Theory*, vol. 29, no. 2, pp. 208–210, 1983.
- [2] J. Bootle, A. Cerulli, P. Chaidos, J. Groth, and C. Petit, "Efficient zero-knowledge arguments for arithmetic circuits in the discrete log setting," Cryptology ePrint Archive, Paper 2016/263, 2016, <https://eprint.iacr.org/2016/263>. [Online]. Available: <https://eprint.iacr.org/2016/263>
- [3] e. a. Breidenbach, L., "Chainlink 2.0: Next steps in the evolution of decentralized oracle networks," <https://research.chain.link/whitepaper-v2.pdf>, 2021.
- [4] V. Buterin, "Sticking to 8192 signatures per slot post-SSF: how and why — ethresear.ch," <https://ethresear.ch/t/sticking-to-8192-signatures-per-slot-post-ssf-how-and-why/17989>, 2023, [Accessed 28-01-2024].
- [5] B. Bünz, J. Bootle, D. Boneh, A. Poelstra, P. Wuille, and G. Maxwell, "Bulletproofs: Short proofs for confidential transactions and more," in *2018 IEEE Symposium on Security and Privacy (SP)*, 2018, pp. 315–334.
- [6] T. D. C. Developers, "Bulletproofs in rust," <https://github.com/dalek-cryptography/bulletproofs>, 2024.
- [7] P. Feldman, "A practical scheme for non-interactive verifiable secret sharing," in *28th Annual Symposium on Foundations of Computer Science (sfcs 1987)*. IEEE, 1987, pp. 427–438.
- [8] A. Fiat and A. Shamir, "How to prove yourself: Practical solutions to identification and signature problems," in *Advances in Cryptology — CRYPTO' 86*, A. M. Odlyzko, Ed. Berlin, Heidelberg: Springer Berlin Heidelberg, 1987, pp. 186–194.
- [9] S. Garg, A. Jain, P. Mukherjee, R. Sinha, M. Wang, and Y. Zhang, "Cryptography with weights: Mpc, encryption and signatures," in *Advances in Cryptology — CRYPTO 2023*, H. Handschuh and A. Lysyanskaya, Eds. Cham: Springer Nature Switzerland, 2023, pp. 295–327.
- [10] D. Grandjean, L. Heimbach, and R. Wattenhofer, "Ethereum proof-of-stake consensus layer: Participation and decentralization," 2023. [Online]. Available: <https://arxiv.org/abs/2306.10777>

- [11] K. Kaya and A. A. Selçuk, "Secret sharing extensions based on the chinese remainder theorem," Cryptology ePrint Archive, Paper 2010/096, 2010, <https://eprint.iacr.org/2010/096>. [Online]. Available: <https://eprint.iacr.org/2010/096>
- [12] M. Mignotte, "How to share a secret," in *Cryptography*, T. Beth, Ed. Berlin, Heidelberg: Springer Berlin Heidelberg, 1983, pp. 371–375.
- [13] B. Schoenmakers, *Verifiable Secret Sharing*. Boston, MA: Springer US, 2005, pp. 645–647. [Online]. Available: https://doi.org/10.1007/0-387-23483-7_452
- [14] A. Shamir, "How to share a secret," *Commun. ACM*, vol. 22, no. 11, p. 612–613, Nov. 1979. [Online]. Available: <https://doi.org/10.1145/359168.359176>
- [15] M. Sharma, "How to Stake Ethereum — investopedia.com," <https://www.investopedia.com/how-to-stake-ethereum-7482623>, 2023, [Accessed 28-01-2024].
- [16] M. Stadler, "Publicly verifiable secret sharing," in *Advances in Cryptology — EUROCRYPT '96*, U. Maurer, Ed. Berlin, Heidelberg: Springer Berlin Heidelberg, 1996, pp. 190–199.

A. Commitments

We will use the formal definition of commitments as below. In general, Pedersen commitments are used in practice in order to use the same discrete log group for commitments and the rest of the system.

Definition 3 (Commitment). *A non-interactive commitment scheme is a pair of randomized polynomial time algorithms (Setup, Com):*

- $Setup(1^\lambda) \rightarrow pp$: Takes as input a security parameter λ and outputs the parameters for the commitment scheme, pp .
- $Com_{pp}(x; r) \rightarrow c$ takes as input a message $x \in \mathcal{M}_{pp}$ and randomness $r \in \mathcal{R}_{pp}$ and outputs a commitment $c \in \mathcal{C}_{pp}$.

For ease of notation we will often omit the subscript pp when the setup step is clear. In general, it is expected that r is drawn uniformly at random from the randomness space \mathcal{R}_{pp} when generating a fresh commitment.

Definition 4 (Binding Commitment). *A commitment scheme is said to be binding if for all PPT adversaries \mathcal{A} there exists a negligible function μ such that:*

$$\Pr \left[\begin{array}{l} Com(x_0; r_0) \\ = Com(x_1; r_1) \\ \wedge x_0 \neq x_1 \end{array} \middle| \begin{array}{l} pp \leftarrow Setup(1^\lambda), \\ x_0, r_0, x_1, r_1 \leftarrow \mathcal{A}(pp) \end{array} \right] \leq \mu(\lambda)$$

Where probability is taken over all random coins of Setup and \mathcal{A} . If $\mu(\lambda) = 0$ then we say that the scheme is perfectly binding.

Definition 5 (Hiding Commitments). *A commitment scheme is said to be hiding if for all PPT adversaries $\mathcal{A}_1, \mathcal{A}_2$ there exists a negligible function μ such that:*

$$\left| \Pr \left[\begin{array}{l} b = b' \\ r \stackrel{\$}{\leftarrow} \mathcal{R}_{pp}, x_0, x_1 \leftarrow \mathcal{A}_1(pp), \\ c \leftarrow Com(x_b; r), b' \leftarrow \mathcal{A}_2 \end{array} \right] - \frac{1}{2} \right| \leq \mu(\lambda)$$

Where probability is taken over b, r , and all random coins of Setup and \mathcal{A} . If $\mu(\lambda) = 0$ then we say that the scheme is perfectly hiding.

a) *Vector Commitments:* A vector commitment scheme is simply a commitment scheme where the message space \mathcal{M} is a vector space, such as \mathbb{Z}_p^n . The definitions for homomorphic, binding, and hiding commitments apply equally to vector commitments.

B. Commitment Wraparound

Definition 6 (Homomorphic Commitments). *A homomorphic commitment scheme is a non-interactive commitment scheme such that $\mathcal{M}_{pp}, \mathcal{R}_{pp}, \mathcal{C}_{pp}$ are all abelian groups, and for all $x, y \in \mathcal{M}_{pp}$ and $r_x, r_y \in \mathcal{R}_{pp}$:*

$$Com(x; r_x) \otimes Com(y; r_y) = Com(x \oplus y; r_x \oplus r_y)$$

Where \otimes denotes the group operation in \mathcal{C}_{pp} and \oplus denotes the group operation in \mathcal{M}_{pp} or \mathcal{R}_{pp} respectively.

a) *Wraparound:* Notice that by the properties of abelian groups, homomorphic commitments over a finite commitment space such that $|\mathcal{C}| = p$ must necessarily wrap around, such that $Com(x + p; r) = Com(x; r)$. This is obvious for Pedersen commitments, but to see that this is true in general, consider a generator $g \in \mathcal{C}$ such that $Com(\mathbb{1}; 0) = g$, for some interpretation of $\mathbb{1}$. For any integer value p we can write $y = \mathbb{1} \cdot p$ where scalar multiplication in this case is interpreted as the group operation repeated p times. We can thus write:

$$\begin{aligned} Com(x + p; r) &= Com(x; r) \otimes Com(\mathbb{1} \cdot p; 0) \\ &= Com(x; r) \otimes g^p \end{aligned}$$

By Fermat's theorem, g^p is the identity element of the group.

As a concrete example, many proofs including Bulletproofs, use Pedersen Commitments, which for a group \mathbb{G} with generators g, h (generated by Setup) of prime order p in which the discrete logarithm problem is hard, are defined as:

$$Com(x; r) := g^x h^r$$

Pedersen commitments are homomorphic, *computationally* binding, and *perfectly* hiding. Note that the discrete logarithm of h with respect to g and vice versa must be unknown to the prover. This notion can be extended to a vector commitment by having setup generate a vector of generators such that:

$$Com(\vec{x}; r) := \vec{g}^{\vec{x}} h^r = h^r \prod_{i=1}^n g_i^{x_i}$$

C. Zero-Knowledge Arguments

Intuitively, a zero-knowledge proof of knowledge is an interactive protocol between a prover P and a verifier V such that on input x the prover proves to the verifier that it knows a witness w that satisfies a relation $\mathcal{R}(x, w)$, without revealing anything else about w . If the proof is sound for only computationally bounded provers, we call it a zero-knowledge argument (ZKA). If the verifier keeps no secrets, then we say the proof (or argument) is public coin. In this paper we consider only public-coin zero-knowledge arguments, and use the following formal definition.

Definition 7 (Zero-Knowledge Arguments). *Let \mathcal{R} be an NP relation. A zero-knowledge argument (ZKA) Π for \mathcal{R} is an interactive protocol between a prover P and a verifier V consisting of k prover messages and $k - 1$ verifier messages. The prover is defined by a family of randomized algorithms $P = \{P_i\}_{i \in [k]}$, while the verifier is defined by a predicate ϕ such that:*

- The prover P on the i -th round takes as input x and the witness w along with all previous challenge messages c_j for $j \in [i - 1]$ and a random input r_i and outputs a message m_i . Written formally:

$$P_i(x, w, \{c_j\}_{j \in [i-1]}; r_i) \rightarrow m_i, \forall i \in [k]$$

- The verifier V on the i -th round samples a random challenge $c_i \xleftarrow{\$} \{0,1\}^\kappa$, $\forall i \in [k-1]$ and sends it to the prover.
- After receiving all k messages from the prover, the verifier outputs $b \leftarrow \phi(x, \{m_i\}_{i \in [k]}, \{c_i\}_{i \in [k-1]})$.

We denote by $\langle P', V' \rangle$ the random variable corresponding to the output b of V' after interacting with P' in the execution of Π , and by $TR(P', V') = (m_1, c_1, \dots, c_{k-1}, m_k)$ the transcript of the execution of Π in which V' interacts with P' . We define the following security notions:

Completeness: We say that Π has perfect completeness if for all $(x, w) \in \mathcal{R}$:

$$\Pr[\langle P(x, w), V(x) \rangle = 1] = 1$$

Computational Soundness: We say that Π has computational soundness if for all $(x, w) \notin \mathcal{R}$ and PPT provers P^* :

$$\Pr[\langle P^*(x, w), V(x) \rangle = 1] \leq \text{negl}(\lambda)$$

Special Honest Verifier Zero-Knowledge (SHVZK): We say that Π has SHVZK if there exists a PPT simulator \mathcal{S} such that for any $(x, w) \in \mathcal{R}$ the simulator \mathcal{S} on input x outputs a transcript that is indistinguishable from the transcript of the honest prover P run with input x and witness w . Written formally:

$$\{TR(P(x, w), V(x))\} \approx_c \{\mathcal{S}(x)\}$$

D. Arithmetic Circuit Proofs

Informally, an arithmetic circuit proof is a zero-knowledge argument that a set of committed values satisfy a given arithmetic circuit. As we discussed in Section II, we will make extensive use of an efficient proof of circuit satisfiability such as the construction given in [2] and refined in [5]. We assume that the commitments and the circuit operate on some field \mathbb{F} , which in practical terms will generally be \mathbb{Z}_p for some prime p .

Consider a set of n values (v_1, \dots, v_n) , with commitment vector $\vec{V} = (V_1, \dots, V_n) \in \mathcal{C}^n$, $V_i = \text{Com}(v_i; r_{v_i})$. We can define an arithmetic circuit by starting with m multiplication gates. We label the inputs to the i -th gate as a_i and b_i , and the output as c_i . We can represent the circuit using the following two sets of equations:

$$a_i \cdot b_i = c_i \quad 1 \leq i \leq m \quad (9)$$

$$\sum_{i=1}^m w_{j,a,i} a_i + \sum_{i=1}^m w_{j,b,i} b_i + \sum_{i=1}^m w_{j,c,i} c_i = \sum_{i=1}^n w_{j,v,i} v_i + k_j \quad 1 \leq j \leq q \quad (10)$$

Where $w_{j,x,i}$ are weights determined by the circuit. We can write this more succinctly by grouping the values into vectors and matrices in the obvious way.

$$\vec{a} \circ \vec{b} = \vec{c} \quad (11)$$

$$\mathbf{W}_a \cdot \vec{a} + \mathbf{W}_b \cdot \vec{b} + \mathbf{W}_c \cdot \vec{c} = \mathbf{W}_v \cdot \vec{v} + \vec{k} \quad (12)$$

We say that a circuit given by $\mathbf{W}_a, \mathbf{W}_b, \mathbf{W}_c, \mathbf{W}_v$ is satisfied by values \vec{v} if there exists $\vec{a}, \vec{b}, \vec{c}$ such that equations 11 and 12 hold. This leads to a natural definition of a circuit proof.

Definition 8 (Arithmetic Circuit Proof). Given a commitment scheme (Setup, Com) an arithmetic circuit proof is a zero-knowledge argument of knowledge for the relation \mathcal{R}_{ckt} :

$$\begin{aligned} \mathcal{R}_{\text{ckt}}((\vec{V} \in \mathcal{C}^n, \mathbf{W}_a, \mathbf{W}_b, \mathbf{W}_c \in \mathbb{F}^{q \times m}, \mathbf{W}_v \in \mathbb{F}^{q \times n}, \\ \vec{k} \in \mathbb{F}^q), (\vec{v} \in \mathbb{F}^n, \vec{a}, \vec{b}, \vec{c} \in \mathbb{F}^m, \vec{r}_v \in \mathcal{R}^n)) \\ := \\ V_i = \text{Com}(v_i, r_{v_i}) \forall i \in [1, n] \\ \wedge \vec{a} \circ \vec{b} = \vec{c} \\ \wedge \mathbf{W}_a \cdot \vec{a} + \mathbf{W}_b \cdot \vec{b} + \mathbf{W}_c \cdot \vec{c} = \mathbf{W}_v \cdot \vec{v} + \vec{k} \quad (13) \end{aligned}$$

Theorem 3. There exists an arithmetic circuit proof protocol Π_{ckt} with perfect completeness, computational soundness, and perfect special honest verifier zero-knowledge for \mathcal{R}_{ckt} . Moreover, Π_{ckt} has communication complexity $O(\log m)$.

We refer the reader to [5] for the security proof.

a) *Notation:* We will generally write **CKT** in place of $(\mathbf{W}_a, \mathbf{W}_b, \mathbf{W}_c, \mathbf{W}_v, \vec{k})$ when the context is clear, implying that the conditions above are satisfied. For example, invoking the circuit proof protocol we will write $\Pi_{\text{ckt}}[(\vec{V}, \mathbf{CKT}); (\vec{v}, \vec{a}, \vec{b}, \vec{c}, \vec{r}_v)]$.

E. Verifiable Secret Sharing

Intuitively, a verifiable secret sharing scheme (VSS) is a secret sharing scheme with added functionality that allows the participants to verify both the dealer actions and the shares received by other participants. Let pp be the parameters of the scheme, including the number of shares n and any parameters needed to determine the set of authorized access sets as $\mathcal{A}_{pp} = \{A \subseteq [n] \mid A \text{ authorized to reconstruct the secret}\}$, and similarly $\bar{\mathcal{A}}_{pp}$ the set of all access sets explicitly unauthorized to reconstruct the secret, where $\mathcal{A}_{pp} \cap \bar{\mathcal{A}}_{pp} = \emptyset$. We will consider only non-interactive VSS, which is composed of two programs:

- 1) $\text{Share}_{pp}(s) \rightarrow ((v_1, \dots, v_n), \pi)$, the sharing program takes as input a secret value s and produces n shares of the secret and a proof π . The proof π is broadcast to all parties, while v_i is given only to party i .
- 2) $\text{Reconstruct}_{pp}(\{(i, v_i \in \mathcal{K})\}_{i \in A}, \pi) \rightarrow s' \in \mathcal{K} \perp$, the reconstruction programs take a set of shares and reconstructs the secret as s' or $\perp \notin \mathcal{K}$.

In practice there will generally also be a verification program for accepting the shares. As this is not needed for the security of our VSS, and should be quite clear from the construction, we omit the formal definition.

Definition 9. We say that a VSS is secure if for any legal parameters pp that contains a security parameter λ , the following properties hold:

a) *Secrecy:* For any unauthorized set $\bar{A} \in \bar{\mathcal{A}}_{pp}$, and any two secrets s, s' , the statistical distance between the distribution of the shares produced by $\text{Share}_{pp}(s)$ and $\text{Share}_{pp}(s')$ is negligible. That is:

$$SD(\{\pi, \{v_i\}_{i \in \bar{A}} | (v_1, \dots, v_n, \pi) \leftarrow \text{Share}_{pp}(s)\}, \{\pi', \{v'_i\}_{i \in \bar{A}} | (v'_1, \dots, v'_n, \pi') \leftarrow \text{Share}_{pp}(s')\}) \leq \text{negl}(\lambda) \quad (14)$$

b) *Correctness*: For any authorized set $A \in \mathcal{A}$, and any secret s , if $(v_1, \dots, v_n, \pi) \leftarrow \text{Share}_{pp}(s)$ then:

$$\text{Reconstruct}_{pp}(\{v_i\}_{i \in A}, \pi) = s$$

c) *Commitment*: For any PPT adversary D^* taking input pp and outputting $\pi, \{v_i\}_{i \in A}, \{v'_i\}_{i \in A'}$ the probability that the two sets of shares reconstruct different secrets is negligible. That is:

$$\Pr[\perp \neq \text{Reconstruct}_{pp}(\{v_i\}_{i \in A}, \pi) \neq \text{Reconstruct}_{pp}(\{v_i\}_{i \in A'}, \pi) \neq \perp] \leq \text{negl}(\lambda) \quad (15)$$

F. Proof of Range Equations

Lemma 1. Let v be an integer, and \mathbb{F} a field of size at least 2. For $n \geq 0 \in \mathbb{N}$, $v \in [0, 2^n]$ if and only if there exists vectors $\vec{a}, \vec{b} \in \mathbb{F}^n$ such that the following equations hold:

$$\sum_{i=1}^n a_i 2^{i-1} = v \quad (16)$$

$$a_i \cdot b_i = 0 \quad 1 \leq i \leq n \quad (17)$$

$$a_i - b_i - 1 = 0 \quad 1 \leq i \leq n \quad (18)$$

Proof. Rearranging (18) and substituting into (17) gives:

$$a_i(a_i - 1) = 0$$

Thus either $a_i = 0$ or $a_i = 1$. Applying this to (16):

$$0 \leq \sum_{i=1}^n a_i 2^{i-1} = v \leq \sum_{i=1}^n 2^{i-1} < 2^n \quad \square$$

G. Base Proof-of-Mod Circuit Equations

Inputs: v, s .

Constants: $p_0, n_1, n_2 \in \mathbb{N}, p, q \in \mathbb{Z}_{p_0}, t \in \mathbb{Z}_p$, such that:

$$\begin{aligned} p_0 &= pq + t \\ 2^{n_1} &\geq p \\ 2^{n_2} &\geq q \end{aligned}$$

Equations for the base proof-of-mod circuit are shown in Figure 4, written in the form required for Π_{ckt} in Section D. The equations in brackets on the right are meant as an intuitive explanation. Equations eliminate k as an intermediate value, and use a_1, \dots, a_{n_2} instead. This construction uses $3n_1 + 3n_2 + 1$ multiplication gates and $5n_1 + 5n_2 + 9$ constraints.

H. ModSolve Function

Let $\text{Decomp}(v, n) \rightarrow \vec{a}, \vec{b}$ be a function that decomposes v into its binary representation in n bits, such that $\langle \vec{a}, \vec{2}^n \rangle = v$, $\vec{a} \circ \vec{b} = \vec{0}^n$, and $\vec{a} - \vec{b} - \vec{1} = \vec{0}^n$, if such a representation is possible, or $a_1 = v, a_2 = \dots = a_n = 0, b_1 = 0, b_2 = \dots = b_n = -1$ otherwise. Algorithm 3

Algorithm 3: ModSolve Function for Base Proof-of-Mod.

Input: $v \in \mathbb{Z}_p, s \in \mathbb{Z}_{p_0}, p_0, p \in \mathbb{N}$, such that $v = s \pmod{p}$.

Output: $\vec{a}, \vec{b}, \vec{c}$ that solve the equations in Figure 4.

- 1 $n_1 \leftarrow \lceil \log_2 p \rceil$;
- 2 $n_2 \leftarrow \lceil \log_2 \lfloor p_0/p \rfloor \rceil$;
- 3 $k \leftarrow (s - v) \cdot p^{-1} \pmod{p_0}$;
- 4 $\vec{a}_1, \vec{b}_1 \leftarrow \text{Decomp}(k)$;
- 5 $\vec{a}_2, \vec{b}_2 \leftarrow \text{Decomp}(q - k)$;
- 6 $\vec{a}_3, \vec{b}_3 \leftarrow \text{Decomp}(v)$;
- 7 $\vec{a}_4, \vec{b}_4 \leftarrow \text{Decomp}(p - v - 1)$;
- 8 $\vec{a}_5, \vec{b}_5 \leftarrow \text{Decomp}(q - k - 1)$;
- 9 $\vec{a}_6, \vec{b}_6 \leftarrow \text{Decomp}(t - v - 1)$;
- 10 $a_7 \leftarrow a_{5,1} - b_{5,1} - 1$;
- 11 $b_7 \leftarrow a_{6,1} - b_{6,1} - 1$;
- 12 $\vec{a} \leftarrow \vec{a}_1 \parallel \vec{a}_6 \parallel a_7$;
- 13 $\vec{b} \leftarrow \vec{b}_1 \parallel \vec{b}_6 \parallel b_7$;
- 14 $\vec{c} \leftarrow \vec{0}^{3n_1 + 3n_2 + 1}$;
- 15 **return** $\vec{a}, \vec{b}, \vec{c}$;

shows the ModSolve function that calculates the wire values that solve the equations in the previous section.

First, we assume a family of secure hash functions \mathcal{H} , from which we select $H \in \mathcal{H}$ for use in the protocol below. Next we assume that the Fiat-Shamir heuristic is applied to the circuit proof protocol using the same family of hash functions, and that the first message from the prover to the verifier includes vector commitments to the wires of the circuit as A, B, C . Let $tr < \Pi_{CKT} > (\vec{V}, \mathbf{CKT}, \vec{v}, \vec{a}, \vec{b}, \vec{c}, \vec{r}_v, A, B, C, r_a, r_b, r_c) \rightarrow \pi$ be a polynomial-time function that produces an accepting transcript π for the resulting non-interactive proof that does not include A, B, C , and let $V_{CKT}(\vec{V}, CKT, A, B, C, \pi_{ckt})$ be a function that verifies transcript π_{ckt} for Π_{CKT} . We can then construct a non-interactive proof-of-mod protocol as follows using the procedures shown in Algorithm 4 and Algorithm 5 for the prover and verifier respectively.

$$\begin{aligned}
\sum_{i=1}^{n_2} p 2^{i-1} a_i &= s - v & (pk = s - v) \\
\sum_{i=1}^{n_2} 2^{i-1} a_i + \sum_{i=1}^{n_2} 2^{i-1} a_{n_2+i} &= q & (k \leq q) \\
\sum_{i=1}^{n_1} 2^{i-1} a_{2n_2+i} &= v & (0 \leq v) \\
\sum_{i=1}^{n_1} 2^{i-1} a_{n_1+2n_2+i} &= p - v - 1 & (v \leq p - 1) \\
\sum_{i=1}^{n_2} 2^{i-1} a_i + \sum_{i=1}^{n_2} 2^{i-1} a_{2n_1+2n_2+i} &= q - 1 & (k \leq q - 1) \\
\sum_{i=1}^{n_1} 2^{i-1} a_{2n_1+3n_2+i} &= t - v - 1 & (v \leq t - v) \\
a_i - b_i - 1 &= 0 & 1 \leq i \leq 2n_1 + 2n_2 \\
\sum_{i=1}^{n_2} z^{i-1} (a_{2n_1+2n_2+i} - b_{2n_1+2n_2+i} - 1) - a_{3n_1+3n_2+1} &= 0 \\
\sum_{i=1}^{n_1} z^{i-1} (a_{2n_1+3n_2+i} - b_{2n_1+3n_2+i} - 1) - b_{3n_1+3n_2+1} &= 0 \\
c_i &= 0 & 1 \leq i \leq 3n_1 + 3n_2 + 1
\end{aligned}$$

Fig. 4: Base Proof-of-Mod Circuit Equations.

Algorithm 4: Non-interactive Prover for Proof-of-Mod.

Input: $p_0, p \in \mathbb{N}$, $a_0, \dots, a_m \in \mathbb{Z}_{p_0}$, $v = \sum_{i=0}^m a_i p_0^i \pmod p \in \mathbb{Z}_p$, $r_v, r_0, \dots, r_m \in \mathcal{R}$, $V = \text{Com}(v; r_v)$, $A_0, \dots, A_m = \text{Com}(a_0; r_0), \dots, \text{Com}(a_m; r_m)$.

Output: Proof π .

- 1 $\vec{a}, \vec{b}, \vec{c} \leftarrow \text{EModSolve}(v, a_0, \dots, a_m, p_0, p)$;
 - 2 $r_a, r_b, r_c \xleftarrow{\$} \mathcal{R}$;
 - 3 $A, B, C \leftarrow \text{Com}(\vec{a}; r_a), \text{Com}(\vec{b}; r_b), \text{Com}(\vec{c}; r_c)$;
 - 4 $z \leftarrow H(A, B, C)$;
 - 5 $\text{CKT} \leftarrow \text{EModCkt}(p_0, p, z)$;
 - 6 $\pi_{\text{ckt}} \leftarrow \text{tr} \langle \Pi_{\text{CKT}} \rangle ((V, A_0, \dots, A_m), \text{CKT}, (v, a_0, \dots, a_m), \vec{a}, \vec{b}, \vec{c}, (r_v, r_0, \dots, r_m), A, B, C, r_a, r_b, r_c)$;
 - 7 **return** $\pi = (A, B, C, \pi_{\text{ckt}})$;
-

Algorithm 5: Non-interactive Verifier for Proof-of-Mod.

Input: $p_0, p \in \mathbb{N}$, Commitments V, A_0, \dots, A_m , Proof π .

Output: Accept or Reject.

- 1 $A, B, C, \pi_{\text{ckt}} \leftarrow \text{Parse}(\pi)$;
 - 2 $z \leftarrow H(A, B, C)$;
 - 3 $\text{CKT} \leftarrow \text{EModCkt}(p_0, p, z)$;
 - 4 **if** $V_{\text{CKT}}((V, A_0, \dots, A_m), \text{CKT}, A, B, C, \pi_{\text{ckt}}) = \text{Accept}$ **then**
 - 5 | **return** *Accept*;
 - 6 **else**
 - 7 | **return** *Reject*;
 - 8 **end**
-

This page is needed for a bug in long table. Please delete this page.

I. Ethereum Staking Data

Entity	ETH Staked	Marketshare	Virtual Shares	WRSS Bits	WRSS Shares
Lido	9,370,076	31.7588	1588	1611	14
Coinbase	4,329,345	14.6738	734	749	7
Binance	1,162,816	3.9412	197	208	2
Kiln	989,472	3.3537	168	178	2
Figment	937,632	3.1780	159	169	2
Rocket Pool	853,934	2.8943	145	155	2
Kraken	825,601	2.7983	140	150	2
Staked.us	666,492	2.2590	113	123	1
OKX	587,649	1.9918	100	109	1
Bitcoin Suisse	541,942	1.8368	92	102	1
Upbit	374,912	1.2707	64	73	1
stakefish	374,752	1.2702	64	73	1
Mantle	338,048	1.1458	57	67	1
DARMA Capital	326,112	1.1053	55	65	1
Blockdaemon	263,712	0.8938	45	54	1
Frax Finance	231,488	0.7846	39	49	1
P2P.org	230,816	0.7823	39	48	1
Swell	182,688	0.6192	31	40	1
ether.fi	167,844	0.5689	28	38	1
Daniel Wang	151,648	0.5140	26	35	1
CoinSpot	137,664	0.4666	23	33	1
Diva (Pre-launch)	125,443	0.4252	21	30	1
Octant	100,000	0.3389	17	26	1
Stader	82,565	0.2798	14	23	1
Stakewise	76,672	0.2599	13	22	1
MyEtherWallet	63,235	0.2143	11	20	1
XHash	60,800	0.2061	10	19	1
imToken	60,128	0.2038	10	19	1
Bitstamp	52,032	0.1764	9	18	1
Revolut	45,152	0.1530	8	17	1
Gate.io	41,536	0.1408	7	16	1
StakeHound	37,504	0.1271	6	15	1
Liquid Collective	35,520	0.1204	6	15	1
Poloniex	31,072	0.1053	5	14	1
RockX	29,216	0.0990	5	14	1
KuCoin	26,464	0.0897	4	14	1
BlockFi	26,112	0.0885	4	13	1
Bitfinex	24,387	0.0827	4	13	1
Stkr (Ankr)	24,104	0.0817	4	13	1
Everstake	22,784	0.0772	4	13	1
Harbour	20,288	0.0688	3	12	1
arthapala.eth	18,816	0.0638	3	12	1
Consensys	18,240	0.0618	3	12	1
Bake	17,408	0.0590	3	12	1
WEX Exchange	16,000	0.0542	3	12	1
conurtrol.eth	15,840	0.0537	3	12	1
StakeWise	13,824	0.0469	2	11	1
Node DAO	12,288	0.0416	2	11	1
Bitpie	11,456	0.0388	2	11	1
HTX	10,368	0.0351	2	11	1
Mercado Bitcoin	10,368	0.0351	2	11	1
Celsius	8,993	0.0305	2	11	1
bitshameddesk.eth	8,640	0.0293	1	10	1

BTC-e	8,416	0.0285	1	10	1
honoraryape.eth	8,000	0.0271	1	10	1
Taylor Gerring	8,000	0.0271	1	10	1
was.eth	7,680	0.0260	1	10	1
Paxos	7,680	0.0260	1	10	1
Swell (Pre-launch)	7,188	0.0244	1	10	1
CoinDCX	7,136	0.0242	1	10	1
Vitalik Buterin	6,976	0.0236	1	10	1
Tranchess	6,976	0.0236	1	10	1
cryptostake.com	6,592	0.0223	1	10	1
guccilorian.eth	5,088	0.0172			
for.eth	5,056	0.0171			
EPotter	4,640	0.0157			
Ebunker	4,640	0.0157			
Sigma Prime Team	4,608	0.0156			
Nimbus Team	4,608	0.0156			
Teku Team	4,608	0.0156			
Prysm Team	4,608	0.0156			
SharedStake	3,584	0.0121			
StaFi	3,456	0.0117			
Redacted Pirex	3,378	0.0114			
MintDice.com	2,688	0.0091			
Bitget	2,688	0.0091			
Bifrost	1,632	0.0055			
DxPool	992	0.0034			
Blox Staking	992	0.0034			
staked.finance	960	0.0033			
Flipside	512	0.0017			
neukind.com	480	0.0016			
Uphold	448	0.0015			
ClayStack	128	0.0004			
SenseiNode	96	0.0003			
Other Solo Stakers	151,143	0.5123			
Unidentified	5,026,342	17.0362			