New Capacity Bounds for PIR on Graph and Multigraph-Based Replicated Storage

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Abstract

In this paper, we study the problem of private information retrieval (PIR) in both graph-based and multigraph-based replication systems, where each file is stored on exactly two servers, and any pair of servers shares at most r files. We derive upper bounds on the PIR capacity for such systems and construct PIR schemes that approach these bounds.

For graph-based systems, we determine the exact PIR capacity for path graphs and improve upon existing results for complete bipartite graphs and complete graphs.

For multigraph-based systems, we propose a PIR scheme that leverages the symmetry of the underlying graph-based construction, yielding a capacity lower bound for such multigraphs. Furthermore, we establish several general upper and lower bounds on the PIR capacity of multigraphs, which are tight in certain cases.

I. INTRODUCTION

Introduced in [1], private information retrieval (PIR) is the problem of retrieving a specific file from a database stored across multiple servers without revealing the identity of the desired file to any of them, in the information-theoretic sense. The goal of PIR is to design retrieval schemes that minimize the total communication cost, i.e., the total number of bits exchanged between the user and the servers. This was the primary performance metric considered in [1] and has remained a central focus in subsequent works; see, for example, [2]–[6].

Given the typical sizes of files users seek to access today, it is reasonable to assume that the size of the retrieved file is significantly larger than the number of bits uploaded by the user, i.e., the queries sent to the servers. This renders the upload cost negligible in comparison to the download cost from the servers [7]. Motivated by this observation, Sun and Jafar studied the PIR problem in [8], where the objective is to minimize the download cost, i.e., the total number of bits sent by the servers.

In their framework, K files are replicated across N non-communicating (non-colluding) servers. The user generates and sends N queries, one to each server. Upon receiving its query, each server truthfully responds with an answer, enabling the user to reconstruct the desired file. The *rate* of a PIR scheme is defined as the number of desired message bits retrieved per downloaded bit, and the supremum of all achievable rates is referred to as the PIR *capacity*.

Building on the characterization of PIR capacity in [8], a wide range of PIR variants have since been explored, motivated by diverse system assumptions and practical considerations. These studies have extended the classical PIR framework to more general and realistic settings. Notable examples include:

- PIR with colluding servers, where subsets of servers may share their received queries in an attempt to infer the user's desired file [9]–[11].
- Coded PIR, where files are not merely replicated but encoded using linear codes and distributed across servers [7], [12]–[15].
- PIR under adversarial models, accounting for threats such as eavesdropping or Byzantine behavior by some servers [16]–[18].
- Symmetric PIR (SPIR), which adds the constraint of database privacy, ensuring the user learns nothing beyond the requested file [19]–[21].
- Weakly private PIR, which allows for limited information leakage and investigates the trade-off between privacy and download efficiency [22]–[26].

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- PIR with side information, including scenarios where the user knows parts of the database in advance [27], [28], as well as cache-aided PIR, which leverages local caching to improve retrieval efficiency [29].
- PIR schemes that explore the trade-off between storage overhead and download cost [30]–[32].

These and many other extensions continue to be active areas of research. For a comprehensive overview of PIR and its numerous variants, we refer the reader to the survey [33].

All of the aforementioned PIR models are considered over either fully replicated storage systems, where each server stores all files, or coded storage systems, where the data is distributed across servers using an error-correcting code.

Although coding techniques for storage systems have seen significant theoretical and practical advances in recent years, many system designers continue to prefer replication over coding due to several advantages, including improved resiliency to data loss, implementation simplicity, ease of file updates, and high availability (see, e.g., [34]–[38]). Moreover, given the vast volumes of data stored in modern applications, fully replicating an entire database across all servers is often impractical. Instead, in most large-scale storage systems, each server stores only a subset of the database (see, e.g., [39]–[41]), and each file is replicated across a small number of servers.

This practical consideration motivated the study of the PIR problem in storage systems with limited replication using a graph-based model in [42]. In this model, a graph with N vertices and K (hyper-)edges represents a storage system, where each vertex corresponds to a server, and each (hyper-)edge represents a file and consists of the set of vertices (i.e., servers) that store it. This model includes the fully replicated storage system as a special case—namely, a replicated storage system represented by a hypergraph with K multiple hyperedges, each of which consists of all the vertices.

As in [8], the PIR problem over replicated storage systems modeled by graphs—referred to as *the PIR problem for graphs*—was defined by Raviv, Tamo and Yaakobi in [42], where they considered the PIR problem for simple graphs[1] under server collusion. Subsequent work in [43] further explored this problem by deriving bounds on the PIR capacity of certain regular graphs, where the PIR capacity of a graph is defined as the PIR capacity of the replicated storage system modeled by that graph. Later in [44], the authors studied the PIR capacity of graphs in the asymptotic regime, where the number of files tends to infinity, while incorporating additional constraints such as server collusion and secure storage.

More recently, [45] established new upper bounds on PIR capacities for several classes of graphs, including complete, star, and bipartite graphs. However, the exact PIR capacity for many of these graph families remains unknown, except for cases with a small number of vertices. For example, the PIR capacity of the star graph is still unknown beyond the case of $K = 4$, which was only recently resolved in [46].

In the same spirit as [42], [43], [45], [46], we further investigate the PIR problem for several important classes of graphs. Moreover, to overcome some of the limitations inherent in the case of simple graphs, we extend the study to multigraph-based storage systems with finite and uniform edge multiplicity $r \geqslant 2$. Specifically, our contributions are summarized as follows:

- For **graph-based** systems, we consider the PIR problem for three classes of graphs: path graphs, complete bipartite graphs, and complete graphs. For path graphs, we show that the PIR capacity of a path graph with N vertices is $2/N$. For complete bipartite graphs, we establish improved upper and lower bounds on the PIR capacity. Finally, for complete graphs, we present a PIR scheme that achieves the capacity for $N = 3$, and attains a higher rate than existing schemes for $N \geqslant 4$.
- For **multigraph-based** systems, we first propose a general PIR scheme construction based on schemes for graph-based systems that satisfies certain symmetry conditions. This construction yields a lower bound on the PIR capacity for certain classes of multigraphs. We also establish several general lower and upper bounds on the PIR capacity of multigraphs, and demonstrate that these bounds are tight for specific classes of multigraphs under certain parameter regimes.

For the reader's convenience, we summarize our results on the PIR capacity for different classes of graphs and compare them with existing results in Table I. Our results for multigraph-based systems are presented in Table II.

The rest of the paper is organized as follows. In Section II, we formally define our problem setting and present preliminary results on the relationships between PIR capacities of graphs and their subgraphs. Section III focuses on establishing PIR capacity bounds for specific classes of graphs. Then, in Section IV, we extend our analysis to derive PIR capacity bounds for general r-multigraphs. Finally, Section V concludes the paper by highlighting some open problems for future research.

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[1]In a simple graph, every edge consists of exactly two vertices, and every pair of vertices defines at most one edge.

Table I
SUMMARY OF RESULTS ON THE PIR CAPACITY OF GRAPHS

Graph Type	Rate of the Best-Known Scheme (Capacity Lower Bound)	Capacity Upper Bound
Path graph $\mathbf{P}_N$	$\frac{2}{N}$ (Construction 1)	$\frac{2}{N}$ (Theorem III.1)
Star graph $\mathbf{S}_{N+1}$	$\frac{1}{2\sqrt{N+1}}$ [45, Theorem 18]	$\frac{1}{\sqrt{2N-\frac{1}{2}}}$ (Theorem III.4)
Complete bipartite graph $\mathbf{K}_{M,N}$	$\frac{1}{2M\sqrt{N}+M}$ (Theorem III.4)	$\frac{1}{\sqrt{2MN}-\frac{M}{2}}$ (Theorem III.4)
Complete graph $\mathbf{K}_N$	$\frac{6}{5-2^{3-N}} \cdot \frac{1}{N}$ (Construction 2)	$\frac{2}{N+1}$ [45, Theorem 9]
General graph	The same as $\mathbf{K}_N$	$\min\left\{\frac{\Delta}{|E(G)|}, \frac{1}{\nu(G)}\right\}$ [45, Theorem 1]

Table II
SUMMARY OF RESULTS ON THE PIR CAPACITY OF r-MULTIGRAPHS

Graph Type	Rate of the Best-Known Scheme (Capacity Lower Bound)	Capacity Upper Bound
Multi-path $\mathbf{P}_N^{(r)}$	$\frac{2}{N} \cdot \left(2 - \frac{1}{2^{r-1}}\right)^{-1}$ (Theorem IV.1)	$\begin{cases} \frac{2}{N} \cdot \left(2 - \frac{1}{2^{r-1}}\right)^{-1}, & N \text{ even;} \\ \frac{2}{N-1} \cdot \left(2 - \frac{1}{2^{r-1}}\right)^{-1}, & N \text{ odd.} \end{cases}$ (Theorem IV.7)
Multi-cycle $\mathbf{C}_N^{(r)}$	$\frac{2}{N+1} \cdot \left(2 - \frac{1}{2^{r-1}}\right)^{-1}$ (Theorem IV.1)	$\begin{cases} \frac{2}{N} \cdot \left(2 - \frac{1}{2^{r-1}}\right)^{-1}, & r \geqslant 2 \text{ (Theorem IV.7)}; \\ \frac{2}{N+1}, & r = 1 \text{ [45, Theorem 9]}. \end{cases}$
Multi-star $\mathbf{S}_{N+1}^{(r)}$	$\frac{2}{N+1} \cdot \left(2 - \frac{1}{2^{r-1}}\right)^{-1}$ (Theorem IV.1)	$\begin{cases} \left(2 - \frac{1}{2^{r-1}}\right)^{-1}, & r \geqslant 2 \text{ (Theorem IV.7)}; \\ \frac{1}{\sqrt{2N-\frac{1}{2}}}, & r = 1 \text{ (Theorem III.4)}. \end{cases}$
Complete multigraph $\mathbf{K}_N^{(r)}$	$\frac{6}{5-2^{3-N}} \cdot \frac{1}{N} \cdot \left(2 - \frac{1}{2^{r-1}}\right)^{-1}$ (Theorem IV.1)	$\frac{1}{N-(N-1)2^{-r}}$ (Theorem IV.8)
General r-multigraph	The same as $\mathbf{K}_N^{(r)}$	$\min\left\{\frac{\Delta}{|E(G)|}, \frac{1}{\nu(G)}\right\} \cdot \left(2 - \frac{1}{2^{r-1}}\right)^{-1}$ (Theorem IV.7)

II. PRELIMINARIES

Throughout the paper, we use the following standard notations. For integers $1 \leqslant m \leqslant n$, let $[m : n] \triangleq \{m, m+1, \ldots, n\}$ and $[n] \triangleq [1 : n]$. For a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and a subset $R \subseteq [n]$, let $\mathbf{x}|_R$ denote the vector obtained by projecting the coordinates of $\mathbf{x}$ onto R. If $R = [m_1 : m_2]$ for some $1 \leqslant m_1 \leqslant m_2 \leqslant n$, we write $\mathbf{x}|_R$ as $\mathbf{x}[m_1 : m_2]$ for convenience. For a subset A of $[n]$ and a positive integer t, we use $\binom{A}{\leqslant t}$ to denote the family of all subsets of A with size at most t, and we use 2^A to denote the family of all subsets of A. We use $H(\cdot)$ to denote the binary entropy function, and $I(\cdot \, ; \cdot)$ to denote the mutual information between two random variables. Finally, we use $G = (V, E)$ to denote the simple graph with a finite vertex set V and edge set $E \subseteq \binom{V}{2}$.

A. Problem Setting

Let $\mathcal{S} = \{S_1, S_2, \ldots, S_N\}$ denote N non-colluding servers and $\mathcal{W} = \{W_1, W_2, \ldots, W_K\}$ denote K independent files. Each $W_i \in \mathbb{F}_2^L$ is a binary vector of length L chosen uniformly at random from all the vectors of $\mathbb{F}_2^L$, hence,

$$H(W_1, \ldots, W_K) = H(W_1) + \cdots + H(W_K) = K \cdot H(W_1) = KL. \tag{1}$$

In the PIR problem, a user privately generates $\theta \in [K]$ and wishes to retrieve W_θ while keeping θ hidden from each server. Let $\mathcal{Q} \triangleq \{Q_i = Q_i(\theta) : i \in [N]\}$ denote the set of all queries generated by the user. Since the user has no information on the content of the files, the queries are independent of them, which means that

$$I(W_1, \ldots, W_K; Q_1, \ldots, Q_N) = 0. \tag{2}$$

Upon receiving its query Q_i from the user, server S_i generates and replies with an answer $A_i = A_i(\theta)$. We denote $\mathcal{W}_i$ as the set of files stored at S_i. Then, A_i is a function of Q_i and $\mathcal{W}_i$, which implies that

$$H(A_i|Q_i, \mathcal{W}_i) = 0, \ \forall \ i \in [N]. \tag{3}$$

A PIR scheme has two basic requirements: *reliability* and *privacy*, formally described as follows.

Reliability: The user should be able to retrieve the desired file W_θ from the received answers A_i with zero probability of error, hence,

$$H(W_\theta|A_{[N]}) = 0. \tag{4}$$

Privacy: Each server learns no information about the desired file index θ. That is, for any $i \in [N]$ and $\theta \in [K]$, it holds that

$$(Q_i(1), A_i(1), \mathcal{W}_i) \sim (Q_i(\theta), A_i(\theta), \mathcal{W}_i), \tag{5}$$

where we use $X \sim Y$ to indicate that the random variables X and Y have the same distribution. Moreover, when θ is chosen uniformly at random from $[K]$, this is equivalent to

$$H(\theta|Q_i, \mathcal{W}_i) = H(\theta) = \log(K). \tag{6}$$

We define the PIR rate of a retrieval scheme T as the ratio between the retrieved file size in bits, and the total number of bits downloaded as answers, i.e.,

$$R_{T,L} = \frac{L}{\sum_{i \in [N]} H(A_i)}. \tag{7}$$

In the sequel, we will omit the subscripts L and T for simplicity.

In this paper, we limit our scope to the PIR problem over *graph-based replication systems*. In such a system, vertices represent servers and (hyper-)edges represent files. A (hyper-)edge is incident to a vertex if a copy of the corresponding file is stored on the corresponding server. Additionally, we also allow the underlying graph to contain multiple (hyper-)edges, and we refer to such systems as *multigraph-based replication systems*. Moreover, a graph/multigraph-based replication system is called an r-replication system for some positive integer r, if each file is replicated r times across r distinct servers. For convenience, we say that T is a PIR scheme for a graph/multigraph G, if T is a retrieval scheme for the PIR problem over the graph-based replication system corresponding to G.

For a 2-replication system with server sets $\mathcal{S}$ and file sets $\mathcal{W}$, we use $G = (\mathcal{S}, \mathcal{W})$ to denote the underlying graph/multigraph with the vertex set $\mathcal{S}$, where a file $W_k \in \mathcal{W}$ is viewed as the edge $\{S_i, S_j\}$ if the file is stored on servers S_i and S_j. The PIR capacity of G is defined as the best possible rate for arbitrarily large file sizes, i.e.,

$$\mathscr{C}(G) = \sup_{T,L} R_{T,L}. \tag{8}$$

Moreover, we say that a PIR scheme for G is *optimal* if its rate equals the capacity $\mathscr{C}(G)$.

By the privacy requirement, one can easily obtain the following result, which will be useful in proving our PIR capacity upper bounds.

Proposition II.1. *[45, Proposition 2] Given a set of file indices $J \subseteq [K]$, we denote $W_J \triangleq \{W_i : i \in J\}$. Then, for any answer A_i, $i \in [N]$, any requested file index $k \in [K]$, and any $J \subseteq [K]$,*

$$H(A_i|Q_i, W_J, \theta = k) = H(A_i|Q_i, W_J). \tag{9}$$

B. PIR Reductions by Graph Decomposition

In this section, we establish a result that relates the PIR capacity of a graph to the PIR capacities of its subgraphs. This result is useful for constructing PIR schemes for specific graph classes in subsequent sections.

Before presenting the result, we introduce the following definitions. Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two graphs. We say that a graph $G = (V, E)$ is a *vertex-disjoint union* of H_1 and H_2 if $V = V_1 \cup V_2$, $E = E_1 \cup E_2$, and $V_1 \cap V_2 = \emptyset$. Similarly, G is called an *edge-disjoint union* of H_1 and H_2 if $V = V_1 \cup V_2$, $E = E_1 \cup E_2$, and $E_1 \cap E_2 = \emptyset$. Clearly, if G is a vertex-disjoint union of H_1 and H_2, then it is also an edge-disjoint union of H_1 and H_2.

Consider the PIR problem for a graph G that is an edge-disjoint union of two subgraphs, H_1 and H_2. Suppose the user desires a file in H_1. Then, one can simply run a PIR scheme T_1 for H_1 to retrieve the file. However, doing so would result in servers receiving queries only related to files in H_1, thereby revealing to the servers storing files in H_2 that the desired file lies in H_1. To preserve privacy, the user can simultaneously send queries regarding files in H_2—according to a PIR scheme T_2 for H_2—while executing T_1. These additional queries are used to retrieve a random file in H_2. Since H_1 and H_2 are edge-disjoint, the queries sent to each server according to T_1 and T_2 are

independent. This results in a PIR scheme for G in which reliability is ensured by T_1, and privacy is maintained jointly by T_1 and T_2. We refer to this as a scheme for G obtained by independently applying T_1 and T_2.

The following result implies that an optimal PIR scheme for a graph G can be obtained by independently applying optimal schemes to each of its connected components, i.e., the set of vertices that are reachable from every other vertex via a path. Consequently, the task of designing optimal schemes for general graphs reduces to separately designing optimal schemes for connected graphs.

Theorem II.2. *If G is an edge-disjoint union of H_1 and H_2, then*

$$\mathscr{C}(G) \geqslant \left(\mathscr{C}(H_1)^{-1} + \mathscr{C}(H_2)^{-1} \right)^{-1}. \tag{10}$$

Furthermore, equality holds in (10) *if H_1 and H_2 are also vertex-disjoint.*

Since the proof of Theorem II.2 is routine and straightforward, we refer the reader to Appendix A.

III. IMPROVED CAPACITY BOUNDS AND SCHEMES FOR PIR OVER GRAPHS

In this section, we study the PIR capacity of several graph classes. First, we determine the PIR capacity of path graphs by establishing an upper bound and constructing an explicit scheme that achieves it. Next, we derive improved upper and lower bounds on the PIR capacity of complete bipartite graphs. Finally, we present a scheme for complete graphs that attains capacity for $N = 3$ and achieves a higher rate than all existing schemes for $N \geqslant 4$.

A. The PIR Capacity of Path Graphs

For $N \geqslant 2$, let $\mathbf{P}_N$ denote the path graph on N vertices, labeled $S_1, \ldots, S_N$. In the replication system based on $\mathbf{P}_N$, we assume without loss of generality that server S_1 stores file W_1, each intermediate server S_i for $i \in [2 : N-1]$ stores files W_{i-1} and W_i, and server S_N stores file W_{N-1}. See Figure 1 for an illustration of the case $N = 3$.
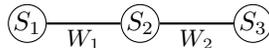

Figure 1. The replication system based on $\mathbf{P}_3$

The following theorem establishes the exact PIR capacity of $\mathbf{P}_N$.

Theorem III.1. *For positive integer $N \geqslant 2$, the path graph $\mathbf{P}_N$ has PIR capacity $\mathscr{C}(\mathbf{P}_N) = \frac{2}{N}$.*

The proof of Theorem III.1 consists of two parts: an upper bound on the capacity, and a matching lower bound achieved via a PIR scheme construction. We begin with the proof of the capacity upper bound.

To prove the capacity upper bound $\mathscr{C}(\mathbf{P}_N) \leqslant \frac{2}{N}$, we need the following results from [45].

Lemma III.2. *[45, Lemma 3] Let $S_i, S_j \in [N]$ be two distinct servers that share the k-th file $W_k \in \mathcal{W}$, then*

$$L \leqslant H(A_i | \mathcal{W} \setminus \{W_k\}, \mathcal{Q}) + H(A_j | \mathcal{W} \setminus \{W_k\}, \mathcal{Q}) \tag{11}$$

$$\leqslant H(A_i) + H(A_j). \tag{12}$$

Theorem III.3. *[45, Theorem 6] Given a PIR scheme for a graph $G = (\mathcal{S}, \mathcal{W})$, then for any server $S_i \in \mathcal{S}$ with degree δ and neighbor set $N(S_i) = \{S_{i_1}, \ldots, S_{i_\delta}\}$, the following holds:*

$$H(A_i | \mathcal{Q}) \geqslant \sum_{j=1}^{\delta} \max \left\{ 0, L - \sum_{\ell=j}^{\delta} H(A_{i_\ell} | \mathcal{Q}) \right\}. \tag{13}$$

Proof of Upper Bound in Theorem III.1: Let T be any feasible PIR scheme for the path graph $\mathbf{P}_N$ with rate R and file length L. It suffices to show that $R \leqslant \frac{2}{N}$. We proceed by analyzing the cases based on the parity of N.

When N is even, assume $N = 2t$ for some positive integer t. Then, we have

$$\sum_{i=1}^{N} H(A_i) = \sum_{j=1}^{t} \left(H(A_{2j-1}) + H(A_{2j}) \right) \tag{14}$$

$$\geqslant tL, \tag{15}$$

where (15) follows from Lemma III.2 and the fact that servers S_{2j-1} and S_{2j} share the file W_{2j-1} for each $1 \leqslant j \leqslant t$. Then, it follows that

$$R = \frac{L}{\sum_{i=1}^{N} H(A_i)} \leqslant \frac{1}{t} = \frac{2}{N}. \tag{16}$$

When N is odd, assume $N = 2t + 1$ for some positive integer t. Similarly, by Lemma III.2 and the fact that servers S_{2j} and S_{2j+1} share the file W_{2j}, we have

$$\sum_{i=1}^{N} H(A_i) = H(A_1) + H(A_2) + H(A_3) + \sum_{j=2}^{t} (H(A_{2j}) + H(A_{2j+1})) \tag{17}$$

$$\geqslant H(A_1) + H(A_2) + H(A_3) + (t-1)L. \tag{18}$$

We claim that $H(A_1) + H(A_2) + H(A_3) \geqslant \frac{3L}{2}$. Then, by (18), it holds that

$$\sum_{i=1}^{N} H(A_i) \geqslant \left(\frac{2t+1}{2}\right) L = \frac{NL}{2}, \tag{19}$$

which implies $R \leqslant \frac{2}{N}$.

Now, we proceed to show that $H(A_1) + H(A_2) + H(A_3) \geqslant \frac{3L}{2}$. Consider the following two cases:

- If $H(A_1) \geqslant \frac{L}{2}$, then, since servers S_2 and S_3 share the file W_2, the claim follows directly by Lemma III.2.
- If $H(A_1) < \frac{L}{2}$, note that server S_2 has the neighbor set $\{S_1, S_3\}$. Thus, by Theorem III.3 and the fact that $H(A_i) \leqslant L$ for every $i \in [N]$, we have

$$H(A_2) \geqslant L - H(A_1) + \max\{0, L - H(A_1) - H(A_3)\} \tag{20}$$

$$\geqslant \begin{cases} 2L - 2H(A_1) - H(A_3), & \text{if } H(A_1) + H(A_3) < L; \\ L - H(A_1), & \text{otherwise.} \end{cases} \tag{21}$$

When $H(A_1) + H(A_3) < L$, using (21) and the assumption that $H(A_1) < \frac{L}{2}$, we get

$$H(A_1) + H(A_2) + H(A_3) \geqslant 2L - H(A_1) \tag{22}$$

$$> \frac{3L}{2}. \tag{23}$$

Otherwise, we have $H(A_1) + H(A_3) \geqslant L$. Then, by (21) and the assumption that $H(A_1) < \frac{L}{2}$, we get

$$H(A_1) + H(A_2) + H(A_3) \geqslant L + H(A_2) \tag{24}$$

$$\geqslant 2L - H(A_1) > \frac{3L}{2}. \tag{25}$$

Therefore, $H(A_1) + H(A_2) + H(A_3) \geqslant \frac{3L}{2}$ holds in both cases. This concludes the proof. ∎

Next, we present a scheme construction for $\mathbf{P}_N$ that achieves the capacity upper bound. The following example provides an illustration of the scheme for the case when $N = 3$.

Example 1. *The replication system based on $\mathbf{P}_3$ consists of three servers, S_1, S_2, and S_3, and two files, $\{W_1, W_2\}$. As illustrated in Figure 1, S_1 stores W_1, S_3 stores W_2, and S_2 stores both W_1 and W_2.*

Suppose that each file consists of two bits, and the user wants to privately retrieve W_θ for some $\theta \in [2]$. To do so, the user first picks a permutation of $[2]$ uniformly at random and applies it to the bit indices of file W_i, for each $i \in [2]$. Denote the resulting permuted file as w_i, $i \in [2]$, and write $w_1 = (a_1, a_2)$ and $w_2 = (b_1, b_2)$. The user then sends queries to retrieve one answer bit from each of S_1, S_2, and S_3, according to Table III.

	S_1	S_2	S_3
$\theta = 1$	a_1	$a_2 + b_2$	b_2
$\theta = 2$	a_1	$a_1 + b_1$	b_2

Table III
ANSWER TABLE FOR $\mathbf{P}_3$.

Clearly, the user can decode the desired file for both cases $\theta = 1$ and $\theta = 2$, and the rate of the scheme is $\frac{2}{3}$. Also, since the bit indices of each file are permuted uniformly at random, the queries appear uniformly distributed regardless of the value of θ. This guarantees privacy.

Construction 1 (A capacity-achieving PIR Scheme for $\mathbf{P}_N$). *Suppose that for each $j \in [N-1]$, the file $W_j = (W_j(1), W_j(2)) \in \mathbb{F}_2^2$ is a binary vector of length 2. The retrieval scheme is described as follows:*

(a) *The user chooses a file index $\theta \in [N-1]$ and $N-1$ permutations $\sigma_j : [2] \to [2]$, independently and uniformly at random, from the set of all permutations on $[2]$. For each $j \in [N-1]$, apply the permutation σ_j to the bits of W_j, and denote the resulting permuted file as:*

$$w_j \triangleq (W_j(\sigma_j(1)), W_j(\sigma_j(2))) = (w_j(1), w_j(2)). \tag{26}$$

(b) *The user sends the query Q_i to each server S_i, $i \in [N]$, defined as:*

$$Q_i = \begin{cases} \sigma_1(1), & \text{if } i = 1; \\ (\sigma_{i-1}(1), \sigma_i(1)), & \text{if } 2 \leqslant i \leqslant \theta; \\ (\sigma_{i-1}(2), \sigma_i(2)), & \text{if } \theta < i \leqslant N-1; \\ \sigma_{N-1}(2), & \text{if } i = N. \end{cases} \tag{27}$$

(c) *Each server S_i, $i \in [N]$ returns the answer $A_i(Q_i)$, defined as:*

$$A_i(Q_i) = \begin{cases} w_1(1), & \text{if } i = 1; \\ w_{i-1}(1) + w_i(1), & \text{if } 2 \leqslant i \leqslant \theta; \\ w_{i-1}(2) + w_i(2), & \text{if } \theta < i \leqslant N-1; \\ w_{N-1}(2), & \text{if } i = N. \end{cases} \tag{28}$$

Proof of Lower Bound in Theorem III.1: It suffices to show that the scheme given in Construction 1 satisfies the privacy and reliability requirements and has the claimed rate.

Privacy: Note that $\{\sigma_j\}_{j \in [N-1]}$ are chosen independently and uniformly at random. Thus, by its definition, the query Q_i sent to server S_i is uniformly distributed over $[2] \times [2]$ for $i \in [2 : N-1]$, and over $[2]$ for $i \in \{1, N\}$. This implies that Q_i reveals no information about θ, which guarantees privacy.

Reliability: By summing all the answers $A_i(Q_i)$ for $1 \leqslant i \leqslant \theta$, the user can retrieve $w_\theta(1)$, as

$$w_\theta(1) = w_1(1) + \sum_{i=2}^{\theta} (w_{i-1}(1) + w_i(1)). \tag{29}$$

Similarly, by summing the remaining answers $A_i(Q_i)$ for $\theta + 1 \leqslant i \leqslant N$, the user can retrieve $w_\theta(2)$, as

$$w_\theta(2) = \sum_{i=\theta+1}^{N-1} (w_{i-1}(2) + w_i(2)) + w_{N-1}(2). \tag{30}$$

Thus, the user can recover W_θ using the permutation σ_θ, which confirms the reliability requirement.

Rate: The size of each file is 2. Since each server returns a single bit, the rate of the scheme is $\frac{2}{N}$. ∎

B. Improved Capacity Bounds for Complete Bipartite Graphs

Let $\mathbf{K}_{M,N}$ denote the complete bipartite graph with two sets of vertices of sizes M and N ($M \leqslant N$), and let $\mathbf{K}_N$ denote the complete graph on N vertices. In [45, Theorem 1], Sadeh, Gu and Tamo established an upper bound on the PIR capacity of general graphs (see also in Table I). As a consequence, they showed that the capacity of $\mathbf{K}_{M,N}$ satisfies

$$\frac{1}{1 - 2^{1-(M+N)}} \cdot \frac{1}{M+N} \leqslant \mathscr{C}(\mathbf{K}_{M+N}) \leqslant \mathscr{C}(\mathbf{K}_{M,N}) \leqslant \frac{1}{M}. \tag{31}$$

In this subsection, we prove the following bounds on $\mathscr{C}(\mathbf{K}_{M,N})$, which improve both the upper and lower bounds compared to those in (31).

Theorem III.4. *The PIR capacity of the complete bipartite graph* $\mathbf{K}_{M,N}$ *satisfies,*

$$\frac{1}{2M\sqrt{N}+M} \leqslant \mathscr{C}(\mathbf{K}_{M,N}) \leqslant \frac{1}{\sqrt{2MN}-\frac{M}{2}}. \tag{32}$$

Remark III.5. *Note that when* $N \geqslant \frac{9M}{8}$, *it holds that*

$$\frac{1}{\sqrt{2MN}-\frac{M}{2}} \leqslant \frac{1}{\sqrt{\frac{9M^2}{4}-\frac{M}{2}}} = \frac{1}{M}. \tag{33}$$

Thus, compared to the upper bound in (31) *by Sadeh, Gu and Tamo [45], the upper bound in Theorem III.4 is tighter when* $N \geqslant \frac{9M}{8}$. *When* $\sqrt{N} \geqslant 5M$, *it holds that*

$$\frac{1}{2M\sqrt{N}+M} \geqslant \frac{1}{\frac{2}{5}N+M} \geqslant \frac{2}{M+N} \tag{34}$$

$$\geqslant \frac{1}{1-2^{1-(M+N)}} \cdot \frac{1}{M+N}. \tag{35}$$

Thus, the lower bound in Theorem III.4 is tighter compared to that in (31) *when* $N \geqslant 25M^2$.

Furthermore, when $M = 1$, *the bound in Theorem III.4 reduces to*

$$\frac{1}{2\sqrt{N}+1} \leqslant \mathscr{C}(\mathbf{K}_{1,N}) \leqslant \frac{1}{\sqrt{2N}-\frac{1}{2}}. \tag{36}$$

The lower bound coincides with that of Theorem 18 in [45]. Moreover, since $\sqrt{2N+1} \leqslant \sqrt{2N}+1$, *the upper bound offers a slight improvement over the upper bound* $\mathscr{C}(\mathbf{K}_{1,N}) \leqslant \frac{1}{\sqrt{2N+1}-2}$ *given by Theorem 8 in [45].*

For the proof of the capacity upper bound in Theorem III.4, we first introduce some necessary results and definitions.

- An *automorphism* of a graph $G = (V, E)$ is a permutation π on the vertex set V that maps edges to edges; that is, two vertices u and v form an edge $\{u, v\} \in E$ if and only if $\{\pi(u), \pi(v)\} \in E$. The automorphism group $\mathrm{Aut}(G)$ of a graph G consists of all its automorphisms.
- A graph G is called *vertex-transitive* if its automorphism group $\mathrm{Aut}(G)$ acts transitively on its vertices, meaning that for any two vertices $u, v \in V(G)$, there exists an automorphism $\pi \in \mathrm{Aut}(G)$ such that $\pi(v) = u$.
- A bipartite graph $G = (V, E)$ with vertex bipartition $\{A, B\}$ is called *vertex-part-transitive* if there exists a subgroup Γ of $\mathrm{Aut}(G)$ that acts transitively on both A and B. This means that for any $a, a' \in A$ and $b, b' \in B$, there exists $\pi \in \Gamma$ such that $\pi(a) = a'$, $\pi(b) = b'$, and $\{a', b'\} \in E$ if and only if $\{a, b\} \in E$.

We also require the following variant of Lemma 11 in [45] for vertex-part-transitive bipartite graphs. For the reader's convenience, we defer its proof to the Appendix B.

Lemma III.6. *Let G be a bipartite graph with vertex bipartition $\{U, V\}$, and suppose G is vertex-part-transitive. Then, for any achievable rate R, there exists a PIR scheme for G with rate R that satisfies:*

$$H(A_i|\mathcal{Q}) = H(A_{i'}|\mathcal{Q}), \quad \forall\, S_i, S_{i'} \in U; \tag{37}$$

$$H(A_j|\mathcal{Q}) = H(A_{j'}|\mathcal{Q}), \quad \forall\, S_j, S_{j'} \in V. \tag{38}$$

Proof of Theorem III.4: We start with the proof of the lower bound. Note that $\mathbf{K}_{M,N}$ can be viewed as the edge-disjoint union of M copies of the star graph $\mathbf{K}_{1,N}$ (also denoted by $\mathbf{S}_{N+1}$) over $N+1$ vertices. Recall that the scheme for $\mathbf{K}_{1,N}$ in [45] achieves a rate of $\frac{1}{2\sqrt{N}+1}$. Thus, the lower bound in Theorem III.4 follows directly by Theorem II.2.

Next, we prove the upper bound.

Consider a PIR scheme for $\mathbf{K}_{M,N}$ with vertex bipartition U and V. Let $S_1, \ldots, S_N$ be the vertices (servers) in U, and $S_{N+1}, \ldots, S_{N+M}$ be the vertices (servers) in V. Since $\mathbf{K}_{M,N}$ is vertex-part-transitive, by Lemma III.6, we have $H(A_i|\mathcal{Q}) = H(A_{i'}|\mathcal{Q})$, for any $i, i' \in [N]$ and $H(A_j|\mathcal{Q}) = H(A_{j'}|\mathcal{Q})$, for any $j, j' \in [N+1 : N+M]$.

Let t be the largest integer in $[N]$ such that $H(A_{N-t+1}|\mathcal{Q}) \leqslant \frac{L}{t}$, and note that $t \geqslant 1$, since $H(A_N|\mathcal{Q}) \leqslant L$. Clearly, by the definition of t, it holds that $H(A_{N-t}|\mathcal{Q}) > \frac{L}{t+1}$. Moreover, since $H(A_i|\mathcal{Q}) = H(A_{i'}|\mathcal{Q})$ for any $i, i' \in [N]$, we can assume that

$$H(A_1|\mathcal{Q}) = H(A_2|\mathcal{Q}) = \cdots = H(A_N|\mathcal{Q}) = \delta L \tag{39}$$

for some $\frac{1}{t+1} < \delta \leqslant \frac{1}{t}$. Then, for every $j \in [N+1 : N+M]$, by Theorem III.3, we have

$$H(A_j|\mathcal{Q}) \geqslant \sum_{i=1}^{N} \max\left\{0, L - \sum_{\ell=i}^{N} H(A_\ell|\mathcal{Q})\right\} \tag{40}$$

$$\geqslant \sum_{i=N-t+1}^{N} \max\left\{0, L - \sum_{\ell=i}^{N} H(A_\ell|\mathcal{Q})\right\} \tag{41}$$

$$= \sum_{i=N-t+1}^{N} \max\left\{0, L - (N-i+1) \cdot \delta L\right\} \tag{42}$$

$$= tL - \frac{t(t+1)}{2} \cdot \delta L \tag{43}$$

where (42) follows by (39), and (43) follows by $\delta \leqslant \frac{1}{t}$. Thus, (43) implies that

$$\sum_{i=1}^{N+M} H(A_i) \geqslant \sum_{i=1}^{N+M} H(A_i|\mathcal{Q}) \tag{44}$$

$$= M \cdot H(A_{N+1}|\mathcal{Q}) + N \cdot H(A_N|\mathcal{Q}) \tag{45}$$

$$\geqslant M\left(tL - \frac{t(t+1)}{2} \cdot \delta L\right) + N\delta L \tag{46}$$

$$= \frac{M\delta L}{2}\left(\frac{2t}{\delta} - t^2 - t\right) + N\delta L \tag{47}$$

$$= -\frac{M\delta L}{2}\left(t - \frac{1}{\delta} + \frac{1}{2}\right)^2 + \frac{M\delta L}{2}\left(\frac{1}{\delta} - \frac{1}{2}\right)^2 + N\delta L \tag{48}$$

$$\geqslant -\frac{M\delta L}{2} \cdot \frac{1}{4} + \frac{M\delta L}{2}\left(\frac{1}{\delta^2} + \frac{1}{4} - \frac{1}{\delta}\right) + N\delta L \tag{49}$$

$$= \frac{ML}{2\delta} + N\delta L - \frac{ML}{2} \tag{50}$$

$$\geqslant \left(\sqrt{2MN} - \frac{M}{2}\right)L, \tag{51}$$

where (49) follows since the function $f(x) = -\left(x - \frac{1}{\delta} + \frac{1}{2}\right)^2$ attains its minimum at the boundaries of the interval and given $t \leqslant \frac{1}{\delta} < t+1$, we have $t - \frac{1}{\delta} + \frac{1}{2} \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, and (51) follows by applying the AM-GM inequality. Therefore, the rate of the scheme satisfies

$$\frac{L}{\sum_{i=1}^{M+N} H(A_i)} \leqslant \frac{L}{\left(\sqrt{2MN} - \frac{M}{2}\right)L} = \frac{1}{\sqrt{2MN} - \frac{M}{2}}, \tag{52}$$

as required. ∎

C. Improved PIR Scheme for Complete Graphs

Let $\mathbf{K}_N$ denote the complete graph over a vertex set of size N. As pointed out in [42] and [45], understanding the PIR capacity of $\mathbf{K}_N$ is of particular importance, as this graph contains the maximum number of files for a given number of servers in graph-based replication systems. Moreover, since any graph G on N vertices is a subgraph of $\mathbf{K}_N$, any PIR scheme for $\mathbf{K}_N$ can be converted into a scheme for G. This implies that $\mathscr{C}(\mathbf{K}_N) \leqslant \mathscr{C}(G)$, which serves as a general lower bound on the PIR capacity of any graph G.

In [45], the PIR capacity of $\mathbf{K}_N$ is shown to be at most $\frac{2}{N+1}$. For the lower bound, it is known from [43] that $\mathscr{C}(\mathbf{K}_3) = \frac{1}{2}$ and $\mathscr{C}(\mathbf{K}_4) \geqslant \frac{3}{10}$. For general N, Sadeh, Gu and Tamo [45] provided a general scheme with rate $\frac{2^{N-1}}{2^{N-1}-1} \cdot \frac{1}{N}$ for any $N \geqslant 2$, which slightly improves upon the previous scheme with rate $\frac{1}{N}$ and demonstrates that the bound of $\frac{1}{N}$ is not optimal.

In this subsection, we show that the scheme by Sadeh, Gu and Tamo [45] can be further modified to achieve a scheme with rate $\frac{6}{5-2^{3-N}} \cdot \frac{1}{N}$. As a consequence, this yields another capacity-achieving scheme that can be

viewed as a reduced version of the one proposed by Banawan and Ulukus in [43] for $\mathbf{K}_3$, while also providing an improvement over their result for $\mathbf{K}_4$.

We begin with the following illustrative example of our PIR scheme for $\mathbf{K}_3$ with the optimal rate $\frac{1}{2}$.

Example 2. *The replication system over $\mathbf{K}_3$ consists of 3 servers, S_1, S_2, S_3 and 3 files. For each $\{i, j\} \subseteq [3]$, let $W_{i,j}$ be the file stored on servers S_i and S_j, as illustrated in Figure 2.*
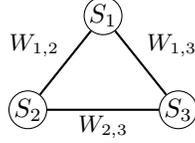

Figure 2. The replication system based on $\mathbf{K}_3$

Suppose that each file consists of $L = 6$ bits, and the user wants to privately retrieve W_θ. To do so, the user privately generates 3 independent permutations of $[L]$, denoted by π_i, $i \in [3]$. For $i \in [L]$, let $a_i \triangleq W_{1,2}(\pi_1(i))$, which represents the $\pi_1(i)$-th bit of $W_{1,2}$. Similarly, define $b_i \triangleq W_{1,3}(\pi_2(i))$ as the $\pi_2(i)$-th bit of $W_{1,3}$, and $c_i \triangleq W_{2,3}(\pi_3(i))$ as the $\pi_3(i)$-th bit of $W_{2,3}$. Then, for every possible θ, the user queries and downloads four bits from each of the servers S_1, S_2, and S_3, as shown in Table IV.

	S_1	S_2	S_3
	a_1	a_3	b_2
	b_6	c_5	c_4
$\theta = (1, 2)$	$a_2 + b_2$	$a_4 + c_4$	$b_5 + c_5$
	$a_5 + b_5$	$a_6 + c_6$	$b_6 + c_6$
	a_6	a_2	b_3
	b_1	c_4	c_5
$\theta = (1, 3)$	$a_2 + b_2$	$a_5 + c_5$	$b_4 + c_4$
	$a_5 + b_5$	$a_6 + c_6$	$b_6 + c_6$
	a_2	a_6	b_5
	b_4	c_1	c_3
$\theta = (2, 3)$	$a_5 + b_5$	$a_2 + c_2$	$b_4 + c_4$
	$a_6 + b_6$	$a_5 + c_5$	$b_6 + c_6$

Table IV
ANSWER TABLE OF OUR PIR SCHEME FOR $\mathbf{K}_3$.

Clearly, the user can decode the desired file in all three cases. For example, consider the case where $\theta = (1, 2)$. According to the answers in Table IV, the user can retrieve all bits in $W_{1,2}$ as follows:

1. *a_1 directly from S_1, and a_3 directly from S_2;*
2. *a_2 by summing $a_2 + b_2$ from S_1 with b_2 from S_3, and a_4 by summing $a_4 + c_4$ from S_2 with c_4 from S_3;*
3. *a_5 by summing $a_5 + b_5$ from S_1 with c_5 from S_2 and $b_5 + c_5$ from S_3, and a_6 by summing $a_6 + c_6$ from S_2 with b_6 from S_1 and $b_6 + c_6$ from S_3.*

Furthermore, since the permutation of bits is unknown to each server, the query for each server appears uniformly distributed regardless of the value of θ. This results in a PIR scheme for $\mathbf{K}_3$ with rate $\frac{6}{4 \times 3} = \frac{1}{2}$, which achieves the capacity upper bound $\frac{2}{N+1}$ for $N = 3$.

Now, we present the formal description of our scheme construction.

Construction 2 (A PIR scheme for $\mathbf{K}_N$). *The system for $\mathbf{K}_N$ consists of N servers, S_i, $i \in [N]$, and $\binom{N}{2}$ files, $W_{i,j}$, $\{i, j\} \in \binom{[N]}{2}$, which is uniquely stored on servers S_i and S_j. Assume further that each file is a binary vector of length $L = 3 \cdot 2^{N-2}$.*

To begin with, the user privately and independently chooses $\binom{N}{2}$ permutations of $[L]$ uniformly at random, denoted by $\pi_{i,j}$, for each $\{i, j\} \in \binom{[N]}{2}$. We denote

$$w_{i,j} = \big(w_{i,j}(1), w_{i,j}(2), \ldots, w_{i,j}(L)\big) \tag{53}$$

with $w_{i,j}(\ell) = W_{i,j}(\pi_{i,j}(\ell))$, as the resulting file after applying permutation $\pi_{i,j}$ to the bits of $W_{i,j}$.

Generating Queries: Assume that $W_{i,i'}$ is the desired file, and let $\mathcal{F}$ be the family of subsets of $[N]$ that contain exactly one of i or i', i.e.,

$$\mathcal{F} = \{P \subseteq [N] : |P \cap \{i, i'\}| = 1\}. \tag{54}$$

Clearly, $|\mathcal{F}| = 2^{N-1}$. Let $\phi : \mathcal{F} \to [2^{N-1}]$ be a fixed bijection. For example, for each $P \in \mathcal{F}$, we set $\phi(P) = \ell$, if P is the ℓ-th set in $\mathcal{F}$ under the lexicographic order. Let $\mathcal{G}$ be the family of all disjoint set pairs (P_1, P_2) satisfying $|P_1| \leqslant |P_2|$ and $P_1 \cup P_2 = [N] \setminus \{i, i'\}$. Clearly, $|\mathcal{G}| = 2^{N-3}$. Let $\varphi : \mathcal{G} \to [2^{N-1} + 1 : 2^{N-1} + 2^{N-3}]$ be a fixed bijection. For example, for each $(P_1, P_2) \in \mathcal{G}$, we set

$$\varphi(P_1, P_2) = \ell + 2^{N-1}, \tag{55}$$

if P_1 is the ℓ-th set in $\binom{[N] \setminus \{i, i'\}}{\leqslant \frac{N-2}{2}}$ under the lexicographic order for some $\ell \in [2^{N-3}]$.

Next, for each server S_j, $j \in [N]$, we construct a map $\sigma_j : 2^{N(j)} \setminus \{\emptyset\} \to [L]$, which is only known to the user, where $N(j) = [N] \setminus \{j\}$ is the neighbor set of S_j in $\mathbf{K}_N$.

For server S_i, let $P \subseteq N(i)$ be a set that contains i' and define

$$\sigma_i(P) = \phi(\{i\} \cup (P \setminus \{i'\})). \tag{56}$$

Note that there are exactly 2^{N-2} sets $P \subseteq N(i)$ containing i'. For $P \subseteq N(i) \setminus \{i'\}$, assume that $|P| \leqslant (N-2)/2$, we define

$$\sigma_i(P) = \sigma_i(N(i) \setminus (P \cup \{i'\})) \tag{57}$$
$$= \varphi(P, N(i) \setminus (P \cup \{i'\})) + 2^{N-3}. \tag{58}$$

Clearly, the σ_i defined above assigns each nonempty subset $P \subseteq N(i)$ a unique index in $[L]$. Similarly, for server $S_{i'}$, we define

$$\sigma_{i'}(P) = \phi(\{i'\} \cup (P \setminus \{i\})) \tag{59}$$

for each $P \subseteq N(i')$ containing i, and

$$\sigma_{i'}(P) = \sigma_{i'}(N(i') \setminus (P \cup \{i\})) \tag{60}$$
$$= \varphi(P, N(i') \setminus (P \cup \{i\})). \tag{61}$$

for each $P \subseteq N(i') \setminus \{i\}$ with $|P| \leqslant (N-2)/2$.

For sever S_j with $j \notin \{i, i'\}$, let P be a subset of $N(j)$ such that $P \in \mathcal{F}$, we define

$$\sigma_j(P) = \phi(P \cup \{j\}). \tag{62}$$

Note that there are exactly 2^{N-2} such sets P. For $P \subseteq N(j)$ such that $\{i, i'\} \subseteq P$, denote $\tilde{P} = \{j\} \cup (P \setminus \{i, i'\})$. Then we define

$$\sigma_j(P) = \begin{cases} \varphi(\tilde{P}, [N] \setminus (\tilde{P} \cup \{i, i'\})), & |\tilde{P}| \leqslant (N-2)/2; \\ \varphi([N] \setminus (\tilde{P} \cup \{i, i'\}), \tilde{P}), & \text{otherwise.} \end{cases} \tag{63}$$

Note that there are exactly 2^{N-3} such sets P. For the remaining $2^{N-3} - 1$ non-empty subsets of $N(j)$, each of which contains neither i nor i', we arbitrarily assign an unused index from $[2^{N-1}]$ to each subset, ensuring that σ_j remains injective.

Next, we describe the bits that the user downloads from each server, which specify the queries each server receives and the answers it provides.

Answers from Servers: The user downloads $2^{N-1} - 1 + 2^{N-3}$ bits from each server S_j: one bit for each nonempty subset of $N(j)$, totaling $2^{N-1} - 1$ bits, and one bit for each set pair (P_1, P_2) in $\mathcal{G}$, totaling 2^{N-3} bits.

Each of these bits is from some sum of the files stored on S_j. Specifically, they are calculated as follows. For a nonempty subset $P \subseteq N(j)$, the corresponding downloaded bit is defined as:

$$D_P^j \triangleq \sum_{\ell \in P} w_{j,\ell}(\sigma_j(P)). \tag{64}$$

When $j = i$, for a set pair (P_1, P_2) in $\mathcal{G}$, the corresponding downloaded bit is defined as:

$$D^i_{(P_1,P_2)} \triangleq \sum_{\ell \in N(i)} w_{i,\ell} \left(\varphi(P_1, P_2) \right). \tag{65}$$

When $j \neq i$, the downloaded bit corresponds to the set pair (P_1, P_2) in $\mathcal{G}$ is defined as:

$$D^j_{(P_1,P_2)} \triangleq \sum_{\ell \in N(j)} w_{j,\ell} \left(\varphi(P_1, P_2) + 2^{N-3} \right). \tag{66}$$

Remark III.7. *Taking the scheme for $\mathbf{K}_3$ in Example 2 as an example, we demonstrate how the user's query bits are generated, specifically the mapping σ_j for each $j \in [3]$.*

Recall that in Example 2, $w_{1,2} = (a_1, \ldots, a_6)$, $w_{1,3} = (b_1, \ldots, b_6)$ and $w_{2,3} = (c_1, \ldots, c_6)$. Assume that $W_{1,2}$ is the desired file. Then, by (54) and the definition of $\mathcal{G}$, we have $\mathcal{F} = \{\{1\}, \{2\}, \{1, 3\}, \{2, 3\}\}$ and $\mathcal{G} = \{(\emptyset, \{3\})\}$.

We set $\phi : \mathcal{F} \to [4]$ as

$$\phi(\{1\}) = 1, \phi(\{1, 3\}) = 2, \phi(\{2\}) = 3, \phi(\{2, 3\}) = 4, \tag{67}$$

and $\varphi : \mathcal{G} \to \{5\}$ as $\varphi(\emptyset, \{3\}) = 5$. Thus, by (56) and (58), σ_1 is defined as

$$\sigma_1(\{2\}) = \phi(\{1\}) = 1, \tag{68}$$
$$\sigma_1(\{2, 3\}) = \phi(\{1, 3\}) = 2, \tag{69}$$
$$\sigma_1(\{3\}) = \varphi(\emptyset, \{3\}) + 1 = 6. \tag{70}$$

By (59) and (61), σ_2 is defined as

$$\sigma_2(\{1\}) = \phi(\{2\}) = 3, \tag{71}$$
$$\sigma_2(\{1, 3\}) = \phi(\{2, 3\}) = 4, \tag{72}$$
$$\sigma_2(\{3\}) = \varphi(\emptyset, \{3\}) = 5, \tag{73}$$

and by (62) and (63), σ_3 is defined as

$$\sigma_3(\{1\}) = \phi(\{1, 3\}) = 2, \tag{74}$$
$$\sigma_3(\{2\}) = \phi(\{2, 3\}) = 4, \tag{75}$$
$$\sigma_3(\{1, 2\}) = \varphi(\emptyset, \{3\}) = 5. \tag{76}$$

Therefore, by (64) and (65), the bits downloaded from S_1 are

$$D^1_{\{2\}} = w_{1,2}(1) = a_1, \ D^1_{\{2,3\}} = w_{1,2}(2) + w_{1,3}(2) = a_2 + b_2, \tag{77}$$
$$D^1_{\{3\}} = w_{1,3}(6) = b_6, \ D^1_{(\emptyset,\{3\})} = w_{1,2}(5) + w_{1,3}(5) = a_5 + b_5; \tag{78}$$

and by (64) and (66), the bits downloaded from S_2 and S_3 are

$$D^2_{\{1\}} = w_{1,2}(3) = a_3, \ D^2_{\{1,3\}} = w_{1,2}(4) + w_{2,3}(4) = a_4 + c_4, \tag{79}$$
$$D^2_{\{3\}} = w_{2,3}(5) = c_5, \ D^2_{(\emptyset,\{3\})} = w_{1,2}(6) + w_{2,3}(6) = a_6 + c_6, \tag{80}$$
$$D^3_{\{1\}} = w_{1,3}(2) = b_2, \ D^3_{\{1,2\}} = w_{1,3}(5) + w_{2,3}(5) = b_5 + c_5, \tag{81}$$
$$D^3_{\{2\}} = w_{2,3}(4) = c_4, \ D^3_{(\emptyset,\{3\})} = w_{1,3}(6) + w_{2,3}(6) = b_6 + c_6. \tag{82}$$

This matches the Table IV.

Theorem III.8. *The scheme by Construction 2 is a PIR scheme for $\mathbf{K}_N$ with rate*

$$\frac{3 \cdot 2^{N-2}}{2^{N-1} + 2^{N-3} - 1} \cdot \frac{1}{N} = \frac{6}{5 - 2^{3-N}} \cdot \frac{1}{N}. \tag{83}$$

Proof: In the following, we show that the scheme by Construction 2 satisfies the reliability and privacy requirements and achieves the claimed rate.

Reliability: We start by showing that the user can compute every bit of $w_{i,i'}$ using the scheme, and therefore retrieve the desired file $W_{i,i'}$.

For $t \in [2^{N-1}]$, let $P \in \mathcal{F}$ be the unique set such that $\phi(P) = t$. Assume that $\{i\} = P \cap \{i, i'\}$. Clearly, it holds that $\{i'\} \cup (P \setminus \{i\}) \in \mathcal{F} \cap 2^{N(i)}$ and $P \setminus \{j\} \in \mathcal{F} \cap 2^{N(j)}$ for each $j \in [N] \setminus \{i, i'\}$. Then, using the downloaded bits from the servers, the user computes the following bit:

$$D^i_{\{i'\} \cup (P \setminus \{i\})} + \sum_{j \in P \setminus \{i\}} D^j_{P \setminus \{j\}}$$

$$= \sum_{\ell \in \{i'\} \cup (P \setminus \{i\})} w_{i,\ell} \left(\sigma_i(\{i'\} \cup (P \setminus \{i\})) \right) + \sum_{j \in P \setminus \{i\}} \sum_{\ell \in P \setminus \{j\}} w_{j,\ell} \left(\sigma_j(P \setminus \{j\}) \right) \tag{84}$$

$$= \sum_{\ell \in \{i'\} \cup (P \setminus \{i\})} w_{i,\ell} \left(\phi(P) \right) + \sum_{j \in P \setminus \{i\}} \sum_{\ell \in P \setminus \{j\}} w_{j,\ell} \left(\phi(P) \right) \tag{85}$$

$$= w_{i,i'}(t) + \sum_{\ell \in P \setminus \{i\}} w_{i,\ell}(t) + \sum_{j \in P \setminus \{i\}} \sum_{\ell \in P \setminus \{j\}} w_{j,\ell}(t) \tag{86}$$

$$= w_{i,i'}(t), \tag{87}$$

where (84) follows by (64), (85) follows by (56) and (62), (86) follows by the assumption that $\phi(P) = t$ and (87) follows since $w_{j,\ell}(t)$ appears exactly twice in the sum for every $\{j, \ell\} \neq \{i, i'\}$. Similarly, if $\{i'\} = P \cap \{i, i'\}$, we can also retrieve $w_{i,i'}(t)$ by computing the bit $D^{i'}_{\{i\} \cup (P \setminus \{i'\})} + \sum_{j \in P \setminus \{i'\}} D^j_{P \setminus \{j\}}$. This shows that the user can retrieve the first 2^{N-1} bits of $w_{i,i'}$.

For $t \in [2^{N-1} + 1 : 2^{N-1} + 2^{N-3}]$, let $(P_1, P_2) \in \mathcal{G}$ be the unique set pair such that $\varphi(P_1, P_2) = t$. By definition, $P_1 \subseteq [N] \setminus \{i, i'\}$ and $|P_1| \leqslant (N-2)/2$. Thus, $P_1 \in 2^{N(i')} \setminus \mathcal{F}$, $(\{i, i'\} \cup P_1 \setminus \{j\}) \subseteq N(j)$ holds for each $j \in P_1$ and $(\{i, i'\} \cup P_2 \setminus \{j\}) \subseteq N(j)$ holds for each $j \in P_2$. Then, using the downloaded bits from the servers, the user computes the following bit:

$$D^i_{(P_1, P_2)} + D^{i'}_{P_1} + D^{i'}_{P_2} + \sum_{j \in P_1} D^j_{\{i, i'\} \cup P_1 \setminus \{j\}} + \sum_{j \in P_2} D^j_{\{i, i'\} \cup P_2 \setminus \{j\}}$$

$$= \sum_{\ell \in N(i)} w_{i,\ell} \left(\varphi(P_1, P_2) \right) + \sum_{\ell \in P_1} w_{i',\ell} \left(\sigma_{i'}(P_1) \right) + \sum_{\ell \in P_2} w_{i',\ell} \left(\sigma_{i'}(P_2) \right)$$

$$+ \sum_{j \in P_1} \sum_{\ell \in \{i, i'\} \cup P_1 \setminus \{j\}} w_{j,\ell} \left(\sigma_j(\{i, i'\} \cup P_1 \setminus \{j\}) \right)$$

$$+ \sum_{j \in P_2} \sum_{\ell \in \{i, i'\} \cup P_2 \setminus \{j\}} w_{j,\ell} \left(\sigma_j(\{i, i'\} \cup P_2 \setminus \{j\}) \right) \tag{88}$$

$$= \sum_{\ell \in N(i)} w_{i,\ell}(t) + \sum_{\ell \in P_1} w_{i',\ell}(t) + \sum_{\ell \in P_2} w_{i',\ell}(t)$$

$$+ \sum_{j \in P_1} \sum_{\ell \in \{i, i'\} \cup P_1 \setminus \{j\}} w_{j,\ell}(t) + \sum_{j \in P_2} \sum_{\ell \in \{i, i'\} \cup P_2 \setminus \{j\}} w_{j,\ell}(t) \tag{89}$$

$$= w_{i,i'}(t) + \sum_{\ell \in [N] \setminus \{i, i'\}} w_{i,\ell}(t) + \sum_{\ell \in [N] \setminus \{i, i'\}} w_{i',\ell}(t)$$

$$+ \sum_{j \in P_1} \sum_{\ell \in \{i, i'\} \cup P_1 \setminus \{j\}} w_{j,\ell}(t) + \sum_{j \in P_2} \sum_{\ell \in \{i, i'\} \cup P_2 \setminus \{j\}} w_{j,\ell}(t)r \tag{90}$$

$$= w_{i,i'}(t), \tag{91}$$

where (88) follows from (64) and (65), (89) follows from (61), (63), and the assumption that $\varphi(P_1, P_2) = t$, and (91) follows since P_1 and P_2 are disjoint sets satisfying $P_1 \cup P_2 = [N] \setminus \{i, i'\}$, which implies $w_{j,\ell}(t)$ appears exactly twice in the sum for every $\{j, \ell\} \neq \{i, i'\}$.

For $t \in [2^{N-1} + 2^{N-3} + 1 : L]$, let $(P_1, P_2) \in \mathcal{G}$ be the unique set pair such that $\varphi(P_1, P_2) = t - 2^{N-3}$. Then,

using the downloaded bits from the servers, the user computes the following bit:

$$D^{i'}_{(P_1,P_2)} + D^i_{P_1} + D^i_{P_2} + \sum_{j \in [N] \setminus \{i,i'\}} D^j_{(P_1,P_2)}$$

$$= \sum_{\ell \in N(i')} w_{i',\ell} \left(\varphi(P_1,P_2) + 2^{N-3} \right) + \sum_{\ell \in P_1} w_{i,\ell} \left(\sigma_i(P_1) \right)$$

$$+ \sum_{\ell \in P_2} w_{i,\ell} \left(\sigma_i(P_2) \right) + \sum_{j \in [N] \setminus \{i,i'\}} \sum_{\ell \in N(j)} w_{j,\ell} \left(\varphi(P_1,P_2) + 2^{N-3} \right) \tag{92}$$

$$= \sum_{\ell \in N(i')} w_{i',\ell}(t) + \sum_{\ell \in P_1} w_{i,\ell}(t) + \sum_{\ell \in P_2} w_{i,\ell}(t)$$

$$+ \sum_{j \in [N] \setminus \{i,i'\}} \sum_{\ell \in N(j)} w_{j,\ell}(t) \tag{93}$$

$$= w_{i,i'}(t) + \sum_{\ell \in [N] \setminus \{i,i'\}} w_{i',\ell}(t) + \sum_{\ell \in [N] \setminus \{i,i'\}} w_{i,\ell}(t)$$

$$+ \sum_{j \in [N] \setminus \{i,i'\}} \sum_{\ell \in N(j)} w_{j,\ell}(t) \tag{94}$$

$$= w_{i,i'}(t), \tag{95}$$

where (92) follows from (64) and (66), (93) follows from (58) and the assumption that $\varphi(P_1,P_2) + 2^{N-3} = t$, and (95) follows since $w_{j,\ell}(t)$ appears exactly twice in the sum for every $\{j,\ell\} \neq \{i,i'\}$.

Privacy: Note that, irrespective of the desired file, by (64), (65), and (66), the downloaded $2^{N-1} + 2^{N-3} - 1$ bits from server S_j always have the following feature: one bit for each non-empty set P of $N(j)$, which has the form

$$\sum_{\ell \in P} W_{j,\ell}(*), \tag{96}$$

and one bit for each set pair $(P_1, P_2) \in \mathcal{G}$, which has the form

$$\sum_{\ell \in N(j)} W_{j,\ell}(*). \tag{97}$$

Moreover, by the definition of $w_{j,\ell}$, from the perspective of server S_j, the index of the bit from the file $W_{j,\ell}$ in each of the sums above is uniformly distributed over all possible choices. Hence, each server learns no information about the file index.

Rate: Each server returns $2^{N-1} + 2^{N-3} - 1$ bits, so the total number of bits downloaded from all servers is $N \cdot (2^{N-1} + 2^{N-3} - 1)$, while the file size is $L = 3 \cdot 2^{N-2}$. Therefore, the rate is

$$\frac{3 \cdot 2^{N-2}}{2^{N-1} + 2^{N-3} - 1} \cdot \frac{1}{N} = \frac{6}{5 - 2^{3-N}} \cdot \frac{1}{N}, \tag{98}$$

as required.

This completes the proof. ∎

IV. CAPACITY BOUNDS AND SCHEME CONSTRUCTIONS FOR PIR OVER MULTIGRAPHS

In this section, we investigate the PIR problem over multigraph-based replication systems. As in prior works, we focus on deriving bounds for the PIR capacity and constructing schemes that approach these bounds. We first propose a general PIR scheme for the r-multigraph extension of any graph that admits a PIR scheme satisfying a certain symmetry condition. This yields a lower bound on the PIR capacity of multigraphs for which such schemes exist. Then, we establish several other lower and upper bounds on the capacity. Our bounds are shown to be tight for certain classes of multigraphs in specific parameter regimes.

For a simple graph $G = (V, E)$ with $|E| = K'$, we denote by $G^{(r)}$ the r-*multigraph extension* of G. That is, $G^{(r)}$ is the multigraph defined on the vertex set V by replacing every edge in G with r parallel edges. Thus, $|E(G^{(r)})| = rK' = K$.

Let $\mathcal{S} = \{S_1, S_2, \ldots, S_N\}$ be a set of N non-colluding servers, and let

$$\mathcal{W}_0 = \{W_{1,0}, W_{2,0}, \ldots, W_{K',0}\} \tag{99}$$

be a collection of K' independent files. This yields a replication system based on the simple graph $G = (\mathcal{S}, \mathcal{W}_0)$. Then, the multigraph $G^{(r)} = (\mathcal{S}, \mathcal{W})$, where

$$\mathcal{W} = \{W_{i,j} : 1 \leqslant i \leqslant K', \, 1 \leqslant j \leqslant r\}, \tag{100}$$

characterizes the 2-replicated system over the server set $\mathcal{S}$ with rK' files indexed by pairs in $[K'] \times [r]$. In this system, each r-subset of files $\{W_{\ell,1}, \ldots, W_{\ell,r}\}$ is stored on servers S_i and S_j if and only if $W_{\ell,0}$ is stored on S_i and S_j in the original graph G. As in Section II-A, we denote by $\mathcal{W}_i$ the set of files stored on server S_i, and by $\mathcal{W}_{i,j}$ the set of r files stored on servers S_i and S_j. Clearly, every $G^{(r)}$ uniquely defines a 2-replication system and the corresponding PIR problem.

For example, Fig. 3 illustrates a system based on the multigraph $\mathbf{P}_3^{(2)}$, i.e., a path graph $\mathbf{P}_3$ on three vertices where each edge is doubled. The system consists of three servers $\mathcal{S} = \{S_1, S_2, S_3\}$ and four files $\mathcal{W} = \{W_{1,1}, W_{1,2}, W_{2,1}, W_{2,2}\}$. Accordingly, server S_1 and S_3 store exactly two files, i.e., $\mathcal{W}_1 = \{W_{1,1}, W_{1,2}\}$, $\mathcal{W}_3 = \{W_{2,1}, W_{2,2}\}$, and server S_2 stores all the four files, i.e., $\mathcal{W}_2 = \mathcal{W}$.
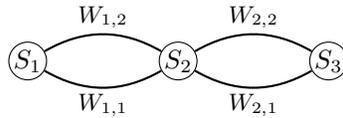

Figure 3. The replication system based on $\mathbf{P}_3^{(2)}$

A. A General PIR Scheme for r-Multigraphs

First, we illustrate the idea of our PIR scheme construction for general r-multigraphs using Example 3. Specifically, based on the scheme for $\mathbf{P}_3$ with rate $\frac{2}{3}$ in Example 1, we construct a scheme for $\mathbf{P}_3^{(2)}$ with rate $\frac{2}{3} \cdot \frac{1}{2 - \frac{1}{2}} = \frac{4}{9}$.

Example 3. *Consider the PIR problem for the multigraph $\mathbf{P}_3^{(2)}$, as illustrated in Figure 3.*

Suppose each file consists of 4 bits, and the user wants to privately retrieve W_θ. The user first privately and independently choose permutations $\sigma_{i,j} : [4] \to [4]$, for $(i,j) \in [K'] \times [r]$, uniformly at random to rearrange the bits of each file, where $K' = r = 2$. Let the permuted file bits be denoted as:

$$\sigma_{1,1}(W_{1,1}) = (a_1, a_2, a_3, a_4), \qquad \sigma_{1,2}(W_{1,2}) = (a_1', a_2', a_3', a_4'), \tag{101}$$
$$\sigma_{2,1}(W_{2,1}) = (b_1, b_2, b_3, b_4), \qquad \sigma_{2,2}(W_{2,2}) = (b_1', b_2', b_3', b_4'). \tag{102}$$

Then, for every possible θ, the user queries and downloads three bits from each server according to Table V.

	S_1	S_2	S_3
$\theta = (1,1)$	a_1	$a_2 + b_2$	b_2
	a_1'	$a_2' + b_2'$	b_2'
	$a_4 + a_2'$	$a_3 + a_1' + b_4 + b_4'$	$b_4 + b_4'$
$\theta = (1,2)$	a_1	$a_2 + b_2$	b_2
	a_1'	$a_2' + b_2'$	b_2'
	$a_2 + a_4'$	$a_1 + a_3' + b_4 + b_4'$	$b_4 + b_4'$
$\theta = (2,1)$	a_1	$a_1 + b_1$	b_2
	a_1'	$a_1' + b_1'$	b_2'
	$a_2 + a_2'$	$a_2 + a_2' + b_4 + b_2'$	$b_3 + b_1'$
$\theta = (2,2)$	a_1	$a_1 + b_1$	b_2
	a_1'	$a_1' + b_1'$	b_2'
	$a_2 + a_2'$	$a_2 + a_2' + b_2 + b_4'$	$b_1 + b_3'$

Table V
ANSWER TABLE FOR $\mathbf{P}_3^{(2)}$.

For every $\theta = (i,j)$, each server responds to the user's query with three answer bits, indicating that the scheme has rate $\frac{4}{3 \times 3} = \frac{4}{9}$. The first answer bits from all three servers match those in the scheme in Example 1, where the

file set is $\{W_{1,1}, W_{2,1}\}$ and the desired file is $W_{i,1}$. Similarly, the second answer bits from all three servers match those in the scheme in Example 1 with the file set $\{W_{1,2}, W_{2,2}\}$ and the desired file $W_{i,2}$. The third answer bits are used to retrieve sums of bits in $W_{i,1}$ and $W_{i,2}$, which are then used to retrieve $W_{i,j}$.

For example, consider the case when $\theta = (1,1)$. The queries sent to server S_1 are:

$$\sigma_{1,1}(1), \sigma_{1,2}(1), \text{ and } (\sigma_{1,1}(4), \sigma_{1,2}(2));\tag{103}$$

the queries sent to server S_2 are:

$$(\sigma_{1,1}(2), \sigma_{2,1}(2)), (\sigma_{1,2}(2), \sigma_{2,2}(2)), \text{ and } (\sigma_{1,1}(3), \sigma_{1,2}(1), \sigma_{2,1}(4), \sigma_{2,2}(4));\tag{104}$$

and the queries sent to server S_3 are:

$$\sigma_{2,1}(2), \sigma_{2,2}(2), \text{ and } (\sigma_{2,1}(4), \sigma_{2,2}(4)).\tag{105}$$

The answer bits from each server are shown in the first row of Table V. Using the first and second answer bits from all three servers, the user can retrieve a_1, a_2 and a'_1, a'_2, respectively. From server S_1, the third answer bit is a new message bit of $W_{1,1}$ added to a known bit of $W_{1,2}$, i.e., $a_4 + a'_2$. From server S_3, the third answer bit is a sum of new message bits of $W_{2,1}$ and $W_{2,2}$, i.e., $b_4 + b'_4$. From S_2, the third answer bit is a sum of bits of all four files stored on S_2, consisting of a new message bit a_3 of $W_{1,1}$, a known bit a'_1, and the third answer bit $b_4 + b'_4$, which is also downloaded from S_3. Therefore, all four bits of $W_\theta = W_{1,1}$ are successfully retrieved.

To understand why privacy holds, note that the query structure seen by each server is identical for all θ. Moreover, the private permutations chosen by the user, hide the actual bit indices from each server.

Next, we introduce our general construction of PIR schemes for r-multigraphs. Before providing the formal description, we need the following definition.

Definition IV.1 (Symmetric Retrieval Property)**.** A graph-based PIR scheme is said to satisfy the symmetric retrieval property (SRP) if, for every possible θ, the number of bits of the file W_θ retrieved from each server storing W_θ is equal. In other words, for every possible θ, it holds that

$$H(A_i|Q_i(\theta), \mathcal{W}_i \setminus \{W_\theta\}) = H(A_j|Q_j(\theta), \mathcal{W}_j \setminus \{W_\theta\}) = \frac{H(W_\theta)}{2},\tag{106}$$

where S_i and S_j are the servers that store W_θ, and $\mathcal{W}_i$ and $\mathcal{W}_j$ denote the sets of files stored at S_i and S_j, respectively.

Theorem IV.1. *Let T denote a scheme for the simple graph G with rate $R(G)$ that satisfies SRP. Then, there exists a scheme $T^{(r)}$ for the r-multigraph extension $G^{(r)}$ of G with rate $R(G) \cdot (2 - 2^{1-r})^{-1}$, which provides the following lower bound on the PIR capacity of $G^{(r)}$:*

$$\mathscr{C}\left(G^{(r)}\right) \geqslant \frac{R(G)}{2 - 2^{1-r}}.\tag{107}$$

Remark IV.2. *The scheme for paths $\mathbf{P}_N$, as described in Construction 1, satisfies SRP since a single bit of the desired file is retrieved from each server storing it. Then, Theorem IV.1 provides a scheme for $\mathbf{P}_N^{(r)}$, leading to*

$$\mathscr{C}\left(\mathbf{P}_N^{(r)}\right) \geqslant \frac{2}{N} \cdot \frac{1}{2 - 2^{1-r}}.\tag{108}$$

The above bound is tight for even N, as demonstrated by a matching upper bound in Theorem IV.7.

Remark IV.3. *For a vertex-transitive graph, by Lemma 11 in [45], there exists a PIR scheme that satisfies SRP, and the bound in (107) holds. In particular, the optimal scheme for cycle graphs $\mathbf{C}_N$ proposed in [43] satisfies SRP, yielding the rate $\frac{2}{N+1}$. Thus, Theorem IV.1 implies the following capacity lower bound for multi-cycles:*

$$\mathscr{C}\left(\mathbf{C}_N^{(r)}\right) \geqslant \frac{2}{N+1} \cdot \frac{1}{2 - 2^{1-r}}.\tag{109}$$

One can also verify that the scheme for complete graphs $\mathbf{K}_N$ in Construction 2 satisfies SRP, which leads to

$$\mathscr{C}\left(\mathbf{K}_N^{(r)}\right) \geqslant \frac{6}{N(5 - 2^{3-N})} \cdot \frac{1}{2 - 2^{1-r}}.\tag{110}$$

Remark IV.4. *The best-known scheme for star graphs* $\mathbf{S}_N$ *(also denoted by* $\mathbf{K}_{1,N-1}$*), as given in [45], does not satisfy SRP. However, a trivial scheme, as described next (see also Scheme 1 in [46]), does.*

Let the central server S_N *store all the files* $\{W_1, \ldots, W_{N-1}\}$ *where each file consists of* 2 *bits, i.e.,* $W_i = (W_i(1), W_i(2))$*. The user privately chooses* $N - 1$ *permutations* $\sigma_i : [2] \to [2]$ *independently and uniformly at random, and applies* σ_i *to* W_i *to obtain* $w_i \triangleq \sigma_i(W_i) = (w_i(1), w_i(2))$*. For any* $\theta \in [N-1]$*, to retrieve* W_θ*, the user downloads the sum of the first bits of all files,* $\sum_{i=1}^{N-1} w_i(1)$*, from* S_N*;* $w_i(1)$ *from* S_i*, where* $i \in [N-1] \setminus \{\theta\}$*; and* $w_\theta(2)$ *from* S_θ*. Clearly, the scheme satisfies SRP and has the rate* $\frac{2}{N}$*. Then, by Theorem IV.1, we have*

$$\mathscr{C}\left(\mathbf{S}_N^{(r)}\right) \geqslant \frac{2}{N} \cdot \frac{1}{2 - 2^{1-r}}. \tag{111}$$

Proof of Theorem IV.1: Let T be a PIR scheme for the graph $G = (\mathcal{S}, \mathcal{W}_0)$, where $\mathcal{W}_0$ is defined in (99). Suppose T admits L' bits per file and D' downloaded bits. Then, $R(G) = \frac{L'}{D'}$. Next, we construct a PIR scheme $T^{(r)}$ for $G^{(r)}$ based on T.

Suppose that each file $W_{s,t}$, $(s,t) \in [K'] \times [r]$, consists of $L = 2^{r-1}L'$ bits in $T^{(r)}$. Before sending out the queries, the user first permutes the L bits of each of the rK' files through a private permutation $\sigma_{s,t} : [L] \to [L]$, chosen independently and uniformly at random. We denote $\sigma_{s,t}(W_{s,t}) = w_{s,t}$. Let $\theta = (i, j)$ be the desired file index, and assume that $W_{i,j}$ is stored on servers S_{n_1} and S_{n_2}. Then, as in the motivating example, the scheme $T^{(r)}$ proceeds in r stages. In each stage, we apply T to several graphs on $\mathcal{S}$, each isomorphic to G, where every edge represents a sum of some of the r files stored on its two associated vertices.

In the following, to simplify the description of the retrieval process, we assume without loss of generality that $\theta = (1, 1)$.

In the first stage, for every $t \in [r]$, we apply T to graph $G_{\{t\}} = (\mathcal{S}, \mathcal{W}_{\{t\}})$, where

$$\mathcal{W}_{\{t\}} \triangleq \{W_{1,t}, W_{2,t}, \ldots, W_{K',t}\}, \tag{112}$$

and retrieve the first L' bits of the file $W_{1,t}$ stored on S_{n_1} and S_{n_2}. This requires rD' downloaded bits in total and results in

$$w_{1,t}[1 : L'], \ \forall \ t \in [r], \tag{113}$$

where $w_{1,1}[1 : L']$ are the desired message bits, and $w_{1,t}[1 : L']$, $t \neq 1$, are the interference bits used for retrieval in the second stage. By the SRP, we can adjust T so that, for each $t \in [r]$, the first half $w_{1,t}[1 : L'/2]$ is retrieved from S_{n_1}, and the latter half $w_{1,t}[L'/2 + 1 : L']$ is retrieved from S_{n_2}.

In the second stage, for every 2-subset $\{j_1, j_2\} \subseteq [r]$, we apply T to the graph $G_{\{j_1,j_2\}} = (\mathcal{S}, \mathcal{W}_{\{j_1,j_2\}})$, where

$$\mathcal{W}_{\{j_1,j_2\}} \triangleq \{W_{i,j_1} + W_{i,j_2} : i \in [K']\},$$

and retrieve L' bits of the file $W_{1,j_1} + W_{1,j_2}$ stored on S_{n_1} and S_{n_2}. Specifically, we perform the following steps:

- If $1 \in \{j_1, j_2\}$, assume without loss of generality that $j_1 = 1$. Then, we apply T to retrieve

$$w_{1,1}[(j_2 - 1)L' + 1 : j_2 L'] + w_{1,j_2}[1 : L']. \tag{114}$$

- If $1 \notin \{j_1, j_2\}$, then we apply T to retrieve

$$w_{1,j_1}[(j_2 - 1)L' + 1 : j_2 L'] + w_{1,j_2}[(j_1 - 1)L' + 1 : j_1 L']. \tag{115}$$

Similarly, by SRP, we can assume that the first $L'/2$ retrieved bits of $w_{1,1} + w_{1,j_2}$ are from S_{n_2}, while the latter $L'/2$ bits are retrieved from S_{n_1}. Moreover, for each $\{j_1, j_2\} \subseteq [r] \setminus \{1\}$, the first $L'/2$ retrieved bits of $w_{1,j_1} + w_{1,j_2}$ are from S_{n_1}, and the latter $L'/2$ bits are retrieved from S_{n_2}.

Consider the bits of $W_{1,j_1} + W_{1,j_2}$ retrieved in the second stage. Since $w_{1,j_2}[1 : L']$ are known by the first stage, we can retrieve $(r - 1)L'$ bits of the form $w_{1,1}[(j_2 - 1)L' + 1 : j_2 L']$, $2 \leqslant j_2 \leqslant r$, for the desired file. Moreover, for each $\{j_1, j_2\} \subseteq [r] \setminus \{1\}$, we also have L' bits from the sum $W_{1,j_1} + W_{1,j_2}$. These $\binom{r-1}{2}L'$ bits are left as interference bits to be used for retrieval in the third stage.

Generally, for $\ell \geqslant 3$ and every $(\ell - 1)$-set $B \subseteq [r] \setminus \{1\}$, we uniquely assign it an integer

$$u_B \in \left[\sum_{s=0}^{\ell-2} \binom{r-1}{s} + 1 : \sum_{s=0}^{\ell-2} \binom{r-1}{s} + \binom{r-1}{\ell-1}\right], \tag{116}$$

and denote $U_B \triangleq [(u_B - 1)L' + 1 : u_B L']$. Then, in the ℓ-th stage, for every ℓ-subset $A \subseteq [r]$, T is applied to graph $G_A = (\mathcal{S}, \mathcal{W}_A)$, where

$$\mathcal{W}_A \triangleq \left\{ \sum_{t \in A} W_{i,t} : \ i \in [K'] \right\}, \tag{117}$$

to retrieve the following L' bits of $\sum_{t \in A} W_{1,t}$:

- If $1 \in A$, then we apply T to retrieve

$$w_{1,1}|_{U_{A \setminus \{1\}}} + \sum_{t \in A \setminus \{1\}} w_{1,t}|_{U_{A \setminus \{1,t\}}}. \tag{118}$$

- If $1 \notin A$, then we apply T to retrieve

$$\sum_{t \in A} w_{1,t}|_{U_{A \setminus \{t\}}}. \tag{119}$$

By SRP, for A containing 1, the first $L'/2$ retrieved bits of $\sum_{t \in A} w_{1,t}$ are from S_{n_2}, while the latter $L'/2$ bits are retrieved from S_{n_1}. Moreover, for A not containing 1, the first $L'/2$ retrieved bits of $\sum_{t \in A} w_{1,t}$ are from S_{n_1}, while the latter $L'/2$ bits are from S_{n_2}.

Using the interference bits of $\sum_{t \in B} w_{1,t}|_{U_{B \setminus \{t\}}}$ from the $(\ell - 1)$-th stage, we can retrieve $\binom{r-1}{\ell-1}L'$ bits of form $w_{1,1}|_{U_B}$, for every $B \subseteq [r] \setminus \{1\}$ of size $\ell - 1$. For each $A \subseteq [r] \setminus \{1\}$ of size ℓ, the L' bits from the sum $\sum_{t \in A} w_{1,t}|_{U_{A \setminus \{t\}}}$ are left as interference bits to be used for retrieval in the next stage.

Reliability: The reliability of $T^{(r)}$ follows directly by that of T.

Privacy: The privacy of $T^{(r)}$ follows from the privacy of T in each stage and the random permutation on the bits of each file in $\mathcal{W}$. Furthermore, the SRP of T guarantees that the $L'/2$ interference bits of $\sum_{t \in B} w_{1,t}|_{U_{B \setminus \{t\}}}$, retrieved from S_{n_1} in the $(\ell - 1)$-th stage for some $(\ell - 1)$-set $B \subseteq [r] \setminus \{1\}$, are used to help retrieve $L'/2$ message bits of $W_{1,1}$ from $w_{1,1}|_{U_B} + \sum_{t \in B} w_{1,t}|_{U_B}$, which is retrieved from S_{n_2} in the ℓ-th stage, and vice versa. This ensures that an equal number of bits of every file is downloaded from the perspective of each server.

Rate: For each $\ell \in [r]$, the user downloads $\binom{r}{\ell}D'$ bits in the ℓ-th stage. The resulting download cost D is therefore,

$$D = \sum_{\ell=1}^{r} \binom{r}{\ell} D' = (2^r - 1)D'. \tag{120}$$

This gives rise to the achievable rate, $\frac{L}{D} = \frac{2^{r-1}L'}{(2^r - 1)D'}$, which is equal to (107). ∎

B. Other Capacity Lower Bounds for r-Multigraphs

By the PIR reduction results in Section II-B, we can obtain the following simple capacity lower bound for general r-multigraphs.

Theorem IV.5. *Let G be a simple graph. Then the PIR capacity of its r-multigraph extension $G^{(r)}$ satisfies*

$$\mathscr{C}(G^{(r)}) \geqslant \frac{\mathscr{C}(G)}{r}. \tag{121}$$

Proof: The proof follows directly from Theorem II.2 by viewing $G^{(r)}$ as an edge-disjoint union of r isomorphic copies of G. ∎

Note that as $r \to \infty$, the lower bound given by Theorem IV.5 approaches 0, making it trivial. To determine the asymptotic behavior of the PIR capacity $\mathscr{C}(G^{(r)})$ for a given graph G, we next present an asymptotic lower bound on $\mathscr{C}(G^{(r)})$, which is a special case of Theorem 1 in [44] by Jia and Jafar.

We define the asymptotic PIR capacity of $G^{(r)} = (\mathcal{S}, \mathcal{W})$ as:

$$\mathscr{C}_\infty \left(G^{(r)} \right) \triangleq \lim_{r \to \infty} \mathscr{C} \left(G^{(r)} \right). \tag{122}$$

By (100), $\mathcal{W}$ can be partitioned into disjoint file subsets as: $\mathcal{W} = \bigcup_{k=1}^{K'} \mathcal{W}_k$, where $\mathcal{W}_k = \{W_{k,1}, W_{k,2}, \ldots, W_{k,r}\}$, for each $k \in [K']$. Using the notation of [44], we denote $\mathcal{R}_k$ as the set of servers storing $\mathcal{W}_k$. We denote $\rho_k \triangleq |\mathcal{R}_k|$ for each $k \in [K']$ and define

$$\rho_{\min} \triangleq \min_{k \in [K']} \rho_k. \tag{123}$$

Since we focus on 2-replicated storage, we have $\rho_k = 2$ for each $k \in [K']$ and $\rho_{\min} = 2$.

During their study of the asymptotic capacity of PIR problems over graphs, Jia and Jafar [44] considered a generalized model incorporating additional privacy and security constraints. Specifically, in their setting, the user's privacy is protected against any set of up to Y colluding servers, and the security of the stored data is protected against any set of up to X colluding servers. In this paper, we focus on the setting that does not involve the additional security constraints or server collusion, hence $X = 0$ and $Y = 1$. Then, the statement of Theorem 1 in [44] adopted to our setting yields the following result.

Theorem IV.6. *[44, Theorem 1] The asymptotic capacity of $G^{(r)}$ satisfies,*

$$\mathscr{C}_\infty \left(G^{(r)} \right) \geqslant \frac{\rho_{\min} - X - Y}{N} = \frac{1}{N}. \tag{124}$$

As we shall see later in Section IV-C, this lower bound is tight for the r-multi-path $\mathbf{P}_N^{(r)}$ and the r-multigraph extension $G^{(r)}$ of any regular graph G. Moreover, the above lower bound is achieved by a simple scheme presented in Section III.B of [44] (see also the scheme provided by Raviv, Tamo and Yaakobi in [42, Section III.A]).

C. Capacity Upper Bounds for PIR over r- Multigraphs

In this subsection, we present two capacity upper bounds for PIR schemes over multigraphs. The first bound applies to general multigraphs and is shown to be optimal for a certain class of graphs. The second bound specifically targets Hamiltonian vertex-transitive graphs and provides a slight improvement over the first. Moreover, both upper bounds are asymptotically tight as $r \to \infty$ by Theorem IV.6.

We begin by recalling some necessary definitions and notations. The *incidence matrix* $I(G)$ of a simple graph $G = (V, E)$ is a $|V(G)| \times |E(G)|$ binary matrix, where the (i, j)-th entry is 1 if and only if the i-th vertex is incident with the j-th edge. A *matching* $M \subseteq E$ of G is a subset of edges such that no two edges share a common vertex. The *matching number* of G, denoted by $\nu(G)$, is the maximum size of a matching in G. A graph G is called *Hamiltonian* if it contains a cycle that visits every vertex in the graph. Moreover, a graph G is called *Hamiltonian vertex-transitive* if it is both Hamiltonian and vertex-transitive.

Next, we formally state our capacity upper bounds.

Theorem IV.7. *Let $G = (\mathcal{S}, \mathcal{W}_0)$ be a graph with maximum degree $\Delta(G)$. Then, the PIR capacity of $G^{(r)}$ satisfies:*

$$\mathscr{C} \left(G^{(r)} \right) \leqslant \min \left(\frac{\Delta(G)}{|E(G)|}, \frac{1}{\nu(G)} \right) \cdot \frac{1}{2 - 2^{1-r}}. \tag{125}$$

Theorem IV.8. *Let $G = (\mathcal{S}, \mathcal{W}_0)$ be a Hamiltonian vertex-transitive graph. Then, the PIR capacity of $G^{(r)}$ satisfies:*

$$\mathscr{C}(G^{(r)}) \leqslant \frac{1}{N - (N-1)2^{-r}}. \tag{126}$$

Remark IV.9. *Theorems IV.7 and IV.8 can be viewed as multigraph extensions of Theorems 1 and 9 in [45], respectively. When $r = 1$, the results of Theorems IV.7 and IV.8 coincide with those of Theorems 1 and 9 in [45].*

Remark IV.10. *For finite r, the capacity upper bound given by Theorem IV.7 is tight for certain graph classes. For example, when $G = \mathbf{P}_N$ and N is even, Theorem IV.7 gives*

$$\mathscr{C} \left(\mathbf{P}_N^{(r)} \right) \leqslant \min \left(\frac{2}{N-1}, \frac{1}{N/2} \right) \cdot \frac{1}{2 - 2^{1-r}} \tag{127}$$

$$= \frac{2}{N} \cdot \frac{1}{2 - 2^{1-r}}. \tag{128}$$

This upper bound is shown to be tight by Construction 1 and Theorem IV.1.

For a regular graph G with N vertices, it always holds that $\frac{\Delta(G)}{|E(G)|} = \frac{2}{N}$. Thus, Theorem IV.7 implies that

$$\mathscr{C}(G^{(r)}) \leqslant \frac{1}{N} \cdot \frac{1}{1 - 2^{-r}} \tag{129}$$

holds for any regular graph G with N vertices. Moreover, since every Hamiltonian vertex-transitive graph is a regular graph, Theorem IV.8 provides a slight improvement over this upper bound. Furthermore, as $r \to \infty$, the upper bounds in both Theorems IV.7 and IV.8 reduce to $\frac{1}{N}$, which is tight by Theorem IV.6.

For the proofs of Theorems IV.7 and IV.8, we need several preliminary results. To begin with, we prove the following multigraph version of Lemma III.2, which plays a crucial role in the upcoming proofs of the capacity upper bounds for multigraphs.

Lemma IV.11. *Let S_i and S_j be two distinct servers that share a set of r files $\mathcal{W}_{i,j} \triangleq \{W_1, W_2, \ldots, W_r\} \subseteq \mathcal{W}$. Then*

$$H(A_i) + H(A_j) \geqslant H(A_i|\mathcal{W} \setminus \mathcal{W}_{i,j}, \mathcal{Q}) + H(A_j|\mathcal{W} \setminus \mathcal{W}_{i,j}, \mathcal{Q}) \tag{130}$$

$$\geqslant \left(2 - \frac{1}{2^{r-1}}\right) L. \tag{131}$$

For the proof of Lemma IV.11, we need a modified version of Lemma 6 in [8], adapted to the 2-replicated PIR system in our setting, rather than the original N-replicated setting. We state and prove this modified version below.

Lemma IV.12. *Let S_i and S_j be two distinct servers that share a set of r files $\mathcal{W}_{i,j} \triangleq \{W_1, W_2, \ldots, W_r\} \subseteq \mathcal{W}$. Then for any $s \in \{2, \ldots, r\}$, it holds that*

$$I(\mathcal{W}_{i,j} \setminus W_{[s-1]}; A_i, A_j|\mathcal{W} \setminus \mathcal{W}_{i,j} \cup W_{[s-1]}, \mathcal{Q}, \theta = s - 1)$$
$$\geqslant \frac{L}{2} + \frac{1}{2} \cdot I(\mathcal{W}_{i,j} \setminus W_{[s]}; A_i, A_j|\mathcal{W} \setminus \mathcal{W}_{i,j} \cup W_{[s]}, \mathcal{Q}, \theta = s), \tag{132}$$

where $W_{[s]} \triangleq \{W_1, W_2, \ldots, W_s\}$.

Proof: Denote $\mathcal{W}^c \triangleq \mathcal{W} \setminus \mathcal{W}_{i,j}$. For any $s \in \{2, 3, \ldots, r\}$, we have

$$2I(\mathcal{W}_{i,j} \setminus W_{[s-1]}; A_i, A_j|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s - 1)$$
$$\geqslant I(\mathcal{W}_{i,j} \setminus W_{[s-1]}; A_i|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s - 1)$$
$$\quad + I(\mathcal{W}_{i,j} \setminus W_{[s-1]}; A_j|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s - 1) \tag{133}$$
$$= H(A_i|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s - 1) - H(A_i|\mathcal{W}, \mathcal{Q}, \theta = s - 1)$$
$$\quad + H(A_j|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s - 1) - H(A_j|\mathcal{W}, \mathcal{Q}, \theta = s - 1) \tag{134}$$
$$= H(A_i|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s) - H(A_i|\mathcal{W}, \mathcal{Q}, \theta = s)$$
$$\quad + H(A_j|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s) - H(A_j|\mathcal{W}, \mathcal{Q}, \theta = s) \tag{135}$$
$$= H(A_i|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s) + H(A_j|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s) \tag{136}$$
$$\geqslant H(A_i|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s) + H(A_j|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s, A_i) \tag{137}$$
$$= H(A_i|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s) - H(A_i|\mathcal{W}, \mathcal{Q}, \theta = s)$$
$$\quad + H(A_j|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s, A_i) - H(A_j|\mathcal{W}, \mathcal{Q}, \theta = s, A_i) \tag{138}$$
$$= I(\mathcal{W}_{i,j} \setminus W_{[s-1]}; A_i, A_j|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s) \tag{139}$$
$$= H(\mathcal{W}_{i,j} \setminus W_{[s-1]}|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s)$$
$$\quad - H(\mathcal{W}_{i,j} \setminus W_{[s-1]}|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s, A_i, A_j) \tag{140}$$
$$= H(\mathcal{W}_{i,j} \setminus W_{[s-1]}|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s)$$
$$\quad - H(\mathcal{W}_{i,j} \setminus W_{[s-1]}|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s, A_{[N]}) \tag{141}$$
$$= L + H(\mathcal{W}_{i,j} \setminus W_{[s]}|\mathcal{W}^c \cup W_{[s]}, \mathcal{Q}, \theta = s)$$
$$\quad - H(\mathcal{W}_{i,j} \setminus W_{[s-1]}|\mathcal{W}^c \cup W_{[s-1]}, \mathcal{Q}, \theta = s, A_{[N]}) \tag{142}$$
$$= L + H(\mathcal{W}_{i,j} \setminus W_{[s]}|\mathcal{W}^c \cup W_{[s]}, \mathcal{Q}, \theta = s)$$
$$\quad - H(\mathcal{W}_{i,j} \setminus W_{[s]}|\mathcal{W}^c \cup W_{[s]}, \mathcal{Q}, \theta = s, A_{[N]}) \tag{143}$$
$$= L + H(\mathcal{W}_{i,j} \setminus W_{[s]}|\mathcal{W}^c \cup W_{[s]}, \mathcal{Q}, \theta = s)$$
$$\quad - H(\mathcal{W}_{i,j} \setminus W_{[s]}|\mathcal{W}^c \cup W_{[s]}, \mathcal{Q}, \theta = s, A_i, A_j) \tag{144}$$
$$= L + I(\mathcal{W}_{i,j} \setminus W_{[s]}; A_i, A_j|\mathcal{W}^c \cup W_{[s]}, \mathcal{Q}, \theta = s), \tag{145}$$

where (135) follows by Proposition II.1, (136) and (138) follow by (3), (141) and (144) follow since by (3), A_ℓ, $\ell \neq i, j$, is a function of $\mathcal{W}^c$ and $\mathcal{Q}$, (142) follows by (1) and (2), and (143) follows by reliability (4). ∎

Now, with the help of Lemma IV.12, we can proceed to the proof of Lemma IV.11.

Proof of Lemma IV.11: We denote $\mathcal{W}^c \triangleq \mathcal{W} \setminus \mathcal{W}_{i,j}$ and Δ as the following difference of entropies:

$$\Delta \triangleq H(\mathcal{W}_{i,j} \mid \mathcal{W}^c, Q, \theta = 1) - H(\mathcal{W}_{i,j} \mid \mathcal{W}^c, \mathcal{Q}, \theta = 1, A_{[N]}). \tag{146}$$

Then,

$$\Delta = H(\mathcal{W}_{i,j}|\mathcal{W}^c, \mathcal{Q}, \theta = 1) - H(\mathcal{W}_{i,j}|\mathcal{W}^c, \mathcal{Q}, \theta = 1, A_i, A_j) \tag{147}$$
$$= I(\mathcal{W}_{i,j}; A_i, A_j|\mathcal{W}^c, \mathcal{Q}, \theta = 1) \tag{148}$$
$$= H(A_i, A_j|\mathcal{W}^c, \mathcal{Q}, \theta = 1) - H(A_i, A_j|\mathcal{W}, \mathcal{Q}, \theta = 1) \tag{149}$$
$$= H(A_i, A_j|\mathcal{W}^c, \mathcal{Q}, \theta = 1) \tag{150}$$
$$\leqslant H(A_i|\mathcal{W}^c, \mathcal{Q}, \theta = 1) + H(A_j|\mathcal{W}^c, \mathcal{Q}, \theta = 1) \tag{151}$$
$$= H(A_i|\mathcal{W}^c, \mathcal{Q}) + H(A_j|\mathcal{W}^c, \mathcal{Q}), \tag{152}$$

where (147) and (150) follow by (3), and (152) follows by Proposition II.1.

On the other hand, we have

$$\Delta = rL - H(\mathcal{W}_{i,j}|\mathcal{W}^c, \mathcal{Q}, \theta = 1, A_i, A_j) \tag{153}$$
$$= rL - (H(W_1|\mathcal{W}^c, \mathcal{Q}, \theta = 1, A_i, A_j) + H(\mathcal{W}_{i,j} \setminus \{W_1\}|\mathcal{W}^c \cup W_1, \mathcal{Q}, \theta = 1, A_i, A_j)) \tag{154}$$
$$= rL - \left(H(W_1|\mathcal{W}^c, \mathcal{Q}, \theta = 1, A_{[N]}) + H(\mathcal{W}_{i,j} \setminus \{W_1\}|\mathcal{W}^c \cup W_1, \mathcal{Q}, \theta = 1, A_i, A_j)\right) \tag{155}$$
$$= rL - H(\mathcal{W}_{i,j} \setminus \{W_1\}|\mathcal{W}^c \cup W_1, \mathcal{Q}, \theta = 1, A_i, A_j) \tag{156}$$
$$= rL - (H(\mathcal{W}_{i,j} \setminus \{W_1\}|\mathcal{W}^c \cup W_1, \mathcal{Q}, \theta = 1) - I(\mathcal{W}_{i,j} \setminus \{W_1\}; A_i, A_j|\mathcal{W}^c \cup W_1, \mathcal{Q}, \theta = 1)) \tag{157}$$
$$= L + I(\mathcal{W}_{i,j} \setminus \{W_1\}; A_i, A_j|\mathcal{W}^c \cup W_1, \mathcal{Q}, \theta = 1), \tag{158}$$

where (153) follows by (1), (2) and (147), (155) follows by (3), (156) follows by reliability (4), and and (158) follow by (1) and (2).

Next, starting from $s = 2$ and applying Lemma IV.12 repeatedly for $s = 3$ to r, we can obtain that

$$I(\mathcal{W}_{i,j} \setminus \{W_1\}; A_i, A_j|\mathcal{W}^c \cup W_1, \mathcal{Q}, \theta = 1)$$
$$\geqslant \frac{L}{2} + \frac{1}{2} \cdot I(\mathcal{W}_{i,j} \setminus \mathcal{W}_{[2]}; A_i, A_j|\mathcal{W}^c \cup \mathcal{W}_{[2]}, \mathcal{Q}, \theta = 2) \tag{159}$$
$$\geqslant \frac{L}{2} + \frac{L}{4} + \frac{1}{4} \cdot I(\mathcal{W}_{i,j} \setminus \mathcal{W}_{[3]}; A_i, A_j|\mathcal{W}^c \cup \mathcal{W}_{[3]}, \mathcal{Q}, \theta = 3) \tag{160}$$
$$\geqslant \cdots$$
$$\geqslant \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{r-1}}\right) L. \tag{161}$$

Finally, the result follows by combining (158) and (161) together. ∎

Now, with the help of Lemma IV.11, we present the proof of Theorem IV.7, which is similar to that of [45, Theorem 1].

Proof of Theorem IV.7: Consider the rate of some PIR scheme T for $G^{(r)}$:

$$R = \frac{L}{\sum_{i \in [N]} H(A_i)} = \frac{1}{\sum_{i \in [N]} (H(A_i)/L)} = \frac{1}{\sum_{i \in [N]} \mu_i} = \frac{1}{\mathbf{1}_N \cdot \mu^T}, \tag{162}$$

where $\mu_i \triangleq \frac{H(A_i)}{L}$ for each $i \in [N]$, $\mu \triangleq (\mu_1, \ldots, \mu_N)$, and $\mathbf{1}_N$ is the all-ones vector of length N. By Lemma IV.11, if servers S_i and S_j are adjacent in $G^{(r)}$, then $\mu_i + \mu_j \geqslant 2 - \frac{1}{2^{r-1}}$. Hence, we can get an upper bound on R via the reciprocal of the optimal value of the following linear program:

$$\min \quad \mathbf{1}_N \cdot \mu^T,$$
$$\text{s.t.} \quad I(G)^T \cdot \mu^T \geqslant \left(2 - \frac{1}{2^{r-1}}\right) \cdot \mathbf{1}_{K'};$$
$$\mu_i \geqslant 0, \ \forall \ i \in [N], \tag{163}$$

where $I(G)$ is the incidence matrix of graph G and $K' = |\mathcal{W}_0|$. Its dual problem is

$$\begin{aligned} \max \quad & \left(2 - \frac{1}{2^{r-1}}\right) \cdot \mathbf{1}_{K'} \cdot \eta^T, \\ \text{s.t.} \quad & I(G) \cdot \eta^T \leqslant \mathbf{1}_N; \\ & \eta_i \geqslant 0, \ \forall \ i \in [K'], \end{aligned} \quad (164)$$

where η is a vector of length K'. By the primal-dual theory, any feasible solution to the dual problem provides a lower bound for the primal problem. Below, we provide two feasible solutions to the dual problem:

(S1) One can easily verify that $\mathbf{v}_1 = \frac{1}{\Delta(G)} \cdot \mathbf{1}_{K'}$ is a feasible solution of the dual problem, where $\Delta(G)$ is the maximum degree of G. Therefore, by

$$\mathbf{1}_{K'} \cdot \mathbf{v}_1^T = \frac{K'}{\Delta(G)}, \quad (165)$$

the rate R is at most $\Delta(G)/K' \cdot \left(2 - 2^{1-r}\right)^{-1}$.

(S2) Let $M \subseteq \mathcal{W}_0$ be a maximum matching of G, i.e., $|M| = \nu(G)$. Let $\mathbf{v}_2 \in \{0,1\}^{K'}$ be the indicator vector of M, i.e., $\mathbf{v}_2(i) = 1$ if and only if the i-th file $W_{i,0}$ is contained in M. Again, it is easy to verify that $\mathbf{v}_2$ is a feasible solution of the dual problem, and therefore, the rate R is at most $1/\nu(G) \cdot \left(2 - 2^{1-r}\right)^{-1}$.

This completes the proof. ■

With the help of Lemma IV.11, the proof of Theorem IV.8 also follows similarly to that of Theorem 9 in [45]. Thus, we do not repeat it in the main text and instead defer it to the Appendix C for readers' convenience.

V. Conclusion and Future Directions

In this paper, we studied the PIR problem over graph- and multigraph-based replication systems with non-colluding servers. We established several upper bounds on the PIR capacity for specific classes of graphs and general r-multigraphs, which were shown to be tight in certain cases. Additionally, we provided constructions of PIR schemes for some graph classes and general r-multigraphs satisfying certain conditions, improving existing PIR schemes and leading to enhanced capacity lower bounds.

Despite these results, the diverse graph structures that can model replication systems in practical scenarios leave many intriguing problems widely open in this topic. Below, we highlight several open questions that we believe will contribute to a deeper understanding of PIR problems in (hyper-)graph- and multigraph-based replication systems:

1) In Section III-A, we showed that the PIR capacity of the path graph $\mathbf{P}_N$ is $\frac{2}{N}$, and in Section III-B, we proved that the PIR capacity of the complete bipartite graph $\mathbf{K}_{M,N}$ satisfies (32). These results, together with the capacity bounds for cycles, stars, and complete graphs from previous works [42], [43], [45], [46], lead us to question whether graphs containing long paths necessarily exhibit smaller PIR capacities, while graphs without long paths tend to have larger PIR capacities. Resolving this question, whether by proving or disproving it, would provide insights into the PIR capacities of general graphs.

2) In Section IV-A, we showed in Theorem IV.1 that for any graph, a PIR scheme can always be constructed for its r-multigraph extension, provided that the original graph admits a PIR scheme satisfying SRP. This raises the question of whether this condition can be removed or relaxed to achieve a more general characterization of the PIR capacities of a graph and its r-multigraph extension. Alternatively, can it be shown that every graph admits a capacity-achieving PIR scheme that satisfies SRP?

3) In Section IV-C, we extended two known PIR capacity upper bounds for graphs, stated as Theorem 1 and Theorem 9 in [45], to the r-multigraph case. However, there is another capacity upper bound, Theorem 6 in [45], which we were unable to generalize. Note that Theorem 6 in [45] was used to establish the capacity upper bounds for $\mathbf{P}_N$ and $\mathbf{K}_{M,N}$ in Section III-A and Section III-B, respectively. We believe an r-multigraph extension of Theorem 6 in [45] could also be helpful for getting better capacity upper bounds for certain r-multigraphs.

4) In the graph- and multigraph-based replication systems considered in this paper, each file is assumed to be stored on exactly two servers, which is a restrictive condition. Therefore, another natural question is whether the results in this paper, as well as those from previous works, can be extended to hypergraph-based replication systems.

APPENDIX A
PROOFS OF THEOREM II.2

Proof of Theorem II.2: Let T_1 and T_2 be PIR schemes for H_1 and H_2, respectively, both with the same file length L and rates R_1 and R_2. Next, by running T_1 and T_2 independently, we construct a PIR scheme for G with rate $\left(R_1^{-1} + R_2^{-1}\right)^{-1}$ and file length L. Then, (10) follows directly from the definition of the PIR capacity.

We assume without loss of generality that $V(G) = V_1 = V_2 = [N]$. Then, we define the PIR scheme for G as follows:

(a) The user chooses a pair of file indices (θ_1, θ_2) from $[|E_1|] \times [|E_2|]$ uniformly at random and conceals the actual desired file index as one of θ_1 or θ_2.

(b) The user sends a query $Q_i(\theta_1, \theta_2) = \left(Q_i^{T_1}(\theta_1), Q_i^{T_2}(\theta_2)\right)$ to each server S_i, $i \in [N]$, where $Q_i^{T_j}(\theta_j)$, $j \in [2]$, represents the query sent to server S_i in the scheme T_j with desired file index θ_j.

(c) Each server S_i, $i \in [N]$ returns the answer A_i, defined as:

$$A_i(\theta_1, \theta_2) = \left(A_i\left(Q_i^{T_1}(\theta_1)\right), A_i\left(Q_i^{T_2}(\theta_2)\right)\right), \tag{166}$$

where $A_i\left(Q_i^{T_j}(\theta_j)\right)$, $j \in [2]$, is the answer from server S_i in the scheme T_j upon receiving the query $Q_i^{T_j}(\theta_j)$.

We now prove that this scheme has the claimed rate and satisfies the privacy and reliability requirements.

Rate: Since T_1 and T_2 have the same file length L and rates R_1 and R_2, respectively, we have

$$\sum_{i \in [N]} H\left(A_i\left(Q_i^{T_j}(\theta_j)\right)\right) = \frac{L}{R_j}, \tag{167}$$

for each $j \in [2]$. Moreover, since T_1 and T_2 are independent schemes and θ_1 and θ_2 are chosen independently at random, it follows that

$$\sum_{i \in [N]} H(A_i) = \sum_{i \in [N]} \sum_{j \in [2]} H\left(A_i\left(Q_i^{T_j}(\theta_j)\right)\right) = \frac{L}{R_1} + \frac{L}{R_2}. \tag{168}$$

Thus, the proposed scheme has rate $\left(R_1^{-1} + R_2^{-1}\right)^{-1}$.

Privacy: For each $i \in [N]$, let $\mathcal{W}_i$ denote the set of files stored on server S_i. For each $j \in [2]$, let $\mathcal{W}_{i,j}$ be the set of files in H_j that are stored on S_i. Clearly, $\mathcal{W}_{i,1}$ and $\mathcal{W}_{i,2}$ are disjoint, and we have $\mathcal{W}_i = \mathcal{W}_{i,1} \cup \mathcal{W}_{i,2}$.

Since T_1 and T_2 satisfy the privacy requirement, by (5), for each $i \in [N]$ and $j \in [2]$, it holds that

$$\left(Q_i^{T_j}(\theta_j), A_i\left(Q_i^{T_j}(\theta_j)\right), \mathcal{W}_{i,j}\right) \sim \left(Q_i^{T_j}(1), A_i\left(Q_i^{T_j}(1)\right), \mathcal{W}_{i,j}\right). \tag{169}$$

This leads to

$$(Q_i(\theta_1, \theta_2), A_i(\theta_1, \theta_2), \mathcal{W}_i) \sim (Q_i(1, 1), A_i(1, 1), \mathcal{W}_i), \tag{170}$$

which implies that the above scheme for G satisfies the privacy requirement.

Reliability: Based on the answers returned from the servers, for each $j \in [2]$, the user can obtain the answer set $\left\{A_i\left(Q_i^{T_j}(\theta_j)\right) : i \in [N]\right\}$. By the reliability of T_1 and T_2, the user can retrieve both the θ_1-th file in H_1 and the θ_2-th file in H_2, this ensures reliability of the proposed scheme.

Finally, we complete the proof for this case by showing why one can assume without loss of generality that schemes T_1 and T_2 have the same file length. Suppose that T_1 and T_2 have different file lengths L_1 and L_2, respectively. Clearly, for any integer $k \geqslant 1$, one can obtain a scheme T_1' for H_1 with the same rate and file length kL_1 by running the scheme T_1 independently k times. Through this approach, we can obtain schemes T_1' and T_2' for H_1 and H_2 with the same file length $L' = \mathrm{lcm}(L_1, L_2)$, respectively, while maintaining their rates.

Next, we show that equality holds in (10) if G is an vertex-disjoint union of H_1 and H_2.

We assume without loss of generality that $V(G) = [N]$, $V_1 = [N_1]$ and $V_2 = [N] \setminus [N_1]$ for some integer $1 \leqslant N_1 < N$. Suppose, towards a contradiction, that there exists a PIR scheme T for G with file length L and rate $R > \left(\mathscr{C}(H_1)^{-1} + \mathscr{C}(H_2)^{-1}\right)^{-1}$. Hence, for a desired file index θ chosen uniformly at random from $[|E(G)|]$, the answer A_i^T from server S_i satisfies

$$\sum_{i \in [N]} H(A_i^T(\theta)) = \frac{L}{R} < L\left(\mathscr{C}(H_1)^{-1} + \mathscr{C}(H_2)^{-1}\right). \tag{171}$$

Meanwhile, by the privacy requirement (5), we know that

$$A_i^T(1) \sim A_i^T(\theta) \tag{172}$$

holds for each $i \in [N]$.

By running the scheme T only over the servers $\{S_1, \ldots, S_{N_1}\}$ and the file set $\bigcup_{i \in [N_1]} \mathcal{W}_i$, we obtain a PIR scheme T_1 for H_1. Specifically, in T_1, for a desired file index θ_1 chosen uniformly at random from $[|E(H_1)|]$, each server S_i, where $i \in [N_1]$, receives a query $Q_i^T(\theta_1)$ and returns the answer $A_i^T(\theta_1)$. Clearly, T_1 has file length L and rate $R_1 \leqslant \mathscr{C}(H_1)$. Hence, the answer $A_i^T(\theta_1)$ from the server S_i satisfies

$$\sum_{i \in [N_1]} H(A_i^T(\theta_1)) = \frac{L}{R_1} \geqslant \frac{L}{\mathscr{C}(H_1)}. \tag{173}$$

Similarly, by running the scheme T only over the servers $\{S_{N_1+1}, \ldots, S_N\}$ and the file set $\bigcup_{i \in [N] \setminus [N_1]} \mathcal{W}_i$, we obtain a PIR scheme T_2 for H_2 with file length L and rate $R_2 \leqslant \mathscr{C}(H_2)$, which leads to

$$\sum_{i \in [N] \setminus [N_1]} H(A_i^T(\theta_2)) = \frac{L}{R_2} \geqslant \frac{L}{\mathscr{C}(H_2)}, \tag{174}$$

for some θ_2 chosen uniformly at random from $[|E(G)|] \setminus [|E(H_1)|]$.

Then, by (172), (173) and (174) together lead to

$$L\left(\mathscr{C}(H_1)^{-1} + \mathscr{C}(H_2)^{-1}\right) \leqslant \sum_{i \in [N_1]} H(A_i^T(\theta_1)) + \sum_{i \in [N] \setminus [N_1]} H(A_i^T(\theta_2)) \tag{175}$$

$$= \sum_{i \in [N]} H(A_i^T(1)), \tag{176}$$

which contradicts (171). This completes the proof. ∎

APPENDIX B
PROOF OF LEMMA III.6

Let T be a PIR scheme for a vertex-part-transitive bipartite graph G with vertex bipartition $\{U, V\}$. Recall that Γ is the subgroup of $\mathrm{Aut}(G)$ that acts transitively on both U and V. Then, given an automorphism $f \in \Gamma$, one can construct the scheme T_f to retrieve the file with index θ as follows.

Suppose W_θ is stored on servers S_u and S_v. Denote $S_{f(i)}$ as the image of S_i after applying f to G and let $f(\theta)$ be the index of the unique file stored on servers $S_{f(u)}$ and $S_{f(v)}$. Then, to server S_i, the user sends the query

$$Q_i^{T_f} \triangleq Q_{f(i)}^T(f(\theta)), \tag{177}$$

and the server replies with the answer:]

$$A_i^{T_f} \triangleq A_{f(i)}^T. \tag{178}$$

According to (177) and (178), the scheme T_f can be viewed as applying the scheme T on the graph $f(G)$ with the desired file $W_{f(\theta)}$. Note that the graph $f(G)$ is simply a relabeling of every vertex (server) S_i of G by $S_{f(i)}$. By the reliability of T, the user can recover the file $W_{f(\theta)}$, which is the unique file stored on $S_{f(u)}$ and $S_{f(v)}$. Since f is just a relabeling of vertices, it follows that this is also the unique file stored on servers S_u and S_v in the graph G, i.e., the file indexed by θ, as required. The privacy requirement of T_f follows since we are simply running the original scheme T, which satisfies privacy. Clearly, the rate of T_f is the same as that of T.

Next, we present the proof of Lemma III.6

Proof of Lemma III.6: To establish a PIR scheme with the desired property, we define a scheme T' by uniformly selecting one of the schemes T_f, for $f \in \Gamma$, and using it to retrieve the desired file as follows:

(a) The user selects a file $\theta \in [K]$ and an automorphism $f \in \Gamma$ independently and uniformly at random.
(b) The user generates the queries $\mathcal{Q}^{T'} \triangleq \mathcal{Q}^{T_f} = \{Q_i^{T_f}, i \in [N]\}$ and sends them to the servers along with the automorphism f.
(c) Each server S_i responds with the answer $A_i^{T_f}$.

Clearly, T' is a scheme with the same rate as T. Let S_i and $S_{i'}$ be two vertices in part U. By the transitivity of Γ, there exists an automorphism $g \in \Gamma$ of G such that $S_{g(i')} = S_i$. Then:

$$H(A_i^{T'}|\mathcal{Q}^{T'}) = \frac{1}{|\Gamma|} \sum_{f \in \Gamma} H(A_i^{T_f}|\mathcal{Q}^{T_f}) \tag{179}$$

$$= \frac{1}{|\Gamma|} \sum_{f \in \Gamma} H(A_{f(i)}^{T}|\mathcal{Q}^{T}) \tag{180}$$

$$= \frac{1}{|\Gamma|} \sum_{f \in \Gamma} H(A_{f(g(i'))}^{T}|\mathcal{Q}^{T}) \tag{181}$$

$$= \frac{1}{|\Gamma|} \sum_{f \in \Gamma} H(A_{f(i')}^{T}|\mathcal{Q}^{T}) \tag{182}$$

$$= \frac{1}{|\Gamma|} \sum_{f \in \Gamma} H(A_{i'}^{T_f}|\mathcal{Q}^{T_f}) \tag{183}$$

$$= H(A_{i'}^{T'}|\mathcal{Q}^{T'}), \tag{184}$$

where equation (179) follows from the fact that $H(X) = \mathbb{E}[-\log(p(X))]$ and that $f \in \Gamma$ is chosen uniformly, and equation (182) follows because if f ranges over all automorphisms in Γ, then so does $f \circ g$. This proves that $H(A_i|\mathcal{Q}) = H(A_{i'}|\mathcal{Q})$ for any $S_i, S_{i'} \in U$.

Similarly, one can show that $H(A_j|\mathcal{Q}) = H(A_{j'}|\mathcal{Q})$ for any $S_j, S_{j'} \in V$. This concludes the proof. ∎

APPENDIX C
PROOF OF THEOREM IV.8

Proof of Theorem IV.8: Let $\mathcal{W}$ denote the set of all files over $G^{(r)}$. Let $(S_1, \mathcal{W}_1, S_2, \mathcal{W}_2, \ldots, S_N, \mathcal{W}_N, S_1)$ be a Hamiltonian r-multicycle in $G^{(r)}$, where $\mathcal{W}_i = \{W_{i,1}, W_{i,2}, \ldots, W_{i,r}\}$ is the set of r files stored on servers S_i and S_{i+1}, modulo N. Assume that each file has length L and $\theta = (N,1)$ is the desired file index. Then, we have

$$L = H(W_{N,1}) \leqslant H(\mathcal{W}_N \mid \mathcal{Q}, \theta = (N,1)) - H(\mathcal{W}_N \mid A_{[N]}, \mathcal{Q}, \theta = (N,1)) \tag{185}$$

$$= I(\mathcal{W}_N; A_{[N]} \mid \mathcal{Q}, \theta = (N,1)) \tag{186}$$

$$= H(A_{[N]} \mid \mathcal{Q}, \theta = (N,1)) - H(A_{[N]} \mid \mathcal{W}_N, \mathcal{Q}, \theta = (N,1)) \tag{187}$$

$$\leqslant \sum_{i \in [N]} H(A_i \mid \mathcal{Q}, \theta = (N,1)) - H(A_{[N]} \mid \mathcal{W}_N, \mathcal{Q}, \theta = (N,1)) \tag{188}$$

$$\leqslant \sum_{i \in [N]} H(A_i) - H(A_{[N]} \mid \mathcal{W}_N, \mathcal{Q}, \theta = (N,1)), \tag{189}$$

where (185) follows since $H(\mathcal{W}_N \mid \mathcal{Q}, \theta = (N,1)) = rL$ and $H(\mathcal{W}_N \mid A_{[N]}, \mathcal{Q}, \theta = (N,1)) \leqslant (r-1)L$ by reliability (4).

Since the proof of Lemma 11 in [45] can be directly extended to r-multigraphs, it follows from Lemma 11 in [45] and Lemma IV.11 that

$$H(A_i \mid \mathcal{W} \setminus \mathcal{W}_i, \mathcal{Q}, \theta = (i,j)) \geqslant (1 - 2^{-r})L \tag{190}$$

holds for each $i \in [N], j \in [r]$. Then, by rearranging inequality (189), we have

$$\sum_{i \in [N]} H(A_i) \geqslant L + H(A_{[N]} \mid \mathcal{W}_N, \mathcal{Q}, \theta = (N, 1)) \tag{191}$$

$$\geqslant L + H(A_{[N-1]} \mid \mathcal{W}_N, \mathcal{Q}, \theta = (N, 1)) \tag{192}$$

$$= L + \sum_{i=1}^{N-1} H(A_i \mid \mathcal{W}_N, A_{[i-1]}, \mathcal{Q}, \theta = (N, 1)) \tag{193}$$

$$\geqslant L + \sum_{i=1}^{N-1} H(A_i \mid \mathcal{W}_N, A_{[i-1]}, \mathcal{W} \setminus \mathcal{W}_i, \mathcal{Q}, \theta = (N, 1)) \tag{194}$$

$$\geqslant L + \sum_{i=1}^{N-1} H(A_i \mid \mathcal{W} \setminus \mathcal{W}_i, \mathcal{Q}, \theta = (N, 1)) \tag{195}$$

$$\geqslant L + (N-1)(1 - 2^{-r})L \tag{196}$$

$$= \left(N - \frac{N-1}{2^r} \right) L. \tag{197}$$

where (193) follows by the chain rule of entropy, (195) follows since the file set $\mathcal{W}_i$ is stored only on servers S_i and S_{i+1} and by (3), the answers $A_{[i-1]}$ are therefore determined by $\mathcal{Q}$ and $\mathcal{W} \setminus \mathcal{W}_i$, and (196) follows by (190).

Finally, the results follows directly by $R = \frac{L}{\sum_{i \in [N]} H(A_i)}$ and (197). ∎

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